

Radial Fuzzy Systems

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Abstract

The class of radial fuzzy systems is introduced. The fuzzy systems in this class use radial functions to implement membership functions of fuzzy sets and exhibit a shape preservation property in antecedents of their rules. The property is called the radial property. It enables the radial fuzzy systems to have their computational model mathematically tractable under both conjunctive and implicative representations of their rule bases. Coherence of radial implicative fuzzy systems is discussed and a sufficient condition for coherence is stated.

Keywords: fuzzy systems, radial functions, coherence

1. Introduction

In applications of fuzzy computing, the notion of a fuzzy system plays a key role [20, 25, 27, 15]. On a general level, the fuzzy system represents a function from an input set into an output set. What is special about the fuzzy system is how this function is implemented.

The implementation draws on the four building blocks of the fuzzy system, namely: *a fuzzifier, a rule base, an inference engine and a defuzzifier*. The fuzzifier and defuzzifier are peripheral blocks. The heart of the fuzzy system comprises the rule base, which stores knowledge incorporated in the fuzzy system, and the inference engine that constitutes a processing unit of the fuzzy system.

The knowledge stored in the rule base is canonically represented by a set of IF-THEN *rules* [5, 2]. Each rule consists of *the antecedent* (the IF part)

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and *the consequent* (the THEN part). The antecedent and consequent are formed by fuzzy sets that are specified on the input or the output set, respectively. The fuzzy sets enable the fuzzy system to incorporate information that is presented linguistically in the way how people describe input-output relations. This ability is the specialty that distinguishes the fuzzy systems from other modeling devices such as neural networks and has brought success to them in several areas. Among them, the area of process control is the most important one [20, 25, 27, 4].

A concrete computational form of the function implemented by the fuzzy system depends on a mathematical representation of the aforementioned building blocks. A certain freedom is present here that leads to the different types of the fuzzy systems. Diversity in shapes of the membership functions carries out a major portion of this freedom. The other source of variability is the way how the rule base of the fuzzy system is represented. The *conjunctive* and the *implicative* representations are the most common [5, 2].

Under the conjunctive representation, individual rules constitute characteristic points in the input-output set. The whole rule base is then understood as the list of these points and the fuzzy system interpolates between them during its computation.

The implicative representation of the rule base is less frequent in applications, but more challenging. In this case, each single rule is understood as an individual condition/restriction imposed on a modeled input-output relation. Different rules correspond to the different restrictions and are required to hold simultaneously. The view of the stored knowledge is logically driven and the related fuzzy relation can be seen as a kind of theory in the logical sense. The issue of *coherence* (logical consistency) of the rule base is critical in this case [7, 4, 23].

Based on these facts, we are interested in the question of assessing the coherence of the implicative fuzzy systems. The assessment should be as comfortable as possible, which means that we only want to check the parameters of the system. We are further interested in a mutual relationship between the implicative and conjunctive representations.

As it is common in science, a certain unifying view on phenomenon of interest is sought. However, a success is typically reached at the cost of a certain simplification and setting up assumptions. The task is to keep the extent of simplification on that level that allows a researcher to theoretically tackle the problem, but still to obtain non-trivial results and gain a broader view of the original problem.

The same approach is adopted in this paper. We restrict ourselves to the class of the so-called *radial fuzzy systems*. In fact, we are going to introduce this class here. The radial fuzzy systems use radial functions to implement fuzzy sets in their rules and exhibit a shape preservation property in the antecedents of their rules. This property enables the radial fuzzy systems to have a mathematically tractable computational model. It is shown that the question of coherence can be quite comfortably answered for this class. On the other hand, the restriction to the radial fuzzy systems is not limiting because this class still contains a rich variety of practically applicable systems including the most commonly used ones.

The paper is organized as follows. The next section discusses motivation for introducing the radial fuzzy systems from the view of a practical application. Sections 3 and 4 review the very basics of fuzzy set and systems theory in order to the paper be self-contained and the notions employed in further sections have been clearly defined. Section 5 introduces the class of the radial fuzzy systems. Section 6 delivers computational models of these systems. Section 7 deals with coherence of the radial implicative fuzzy systems. Conclusions are given in Section 8 along with the directions of future research.

2. Motivation

When assessing computational capabilities of the fuzzy system, internal consistency of its rule base is a highly desirable property. Consistency refers to the presence of non-contradictory rules in the rule base. It is the desirable feature because conflicting rules might have a fatal impact on the final computation of the system as the following canonical example shows.

Suppose that the fuzzy system is used for car navigation and its rule base consist only of two rules. The input set is the set of possible locations for a frontal obstacle that are measured in angles of deviation from the vertical axis. The output set is the set of actions expressed as angles of future navigation. Both sets are equipped with the following fuzzy sets: *directly ahead* referring to angles about 0° , *go left* to angles about -30° and *go right* to angles about $+30^\circ$, see Fig. 1. The rules are the following:

- 1. IF *an obstacle is directly ahead* THEN *go left*
- 2. IF *an obstacle is directly ahead* THEN *go right*

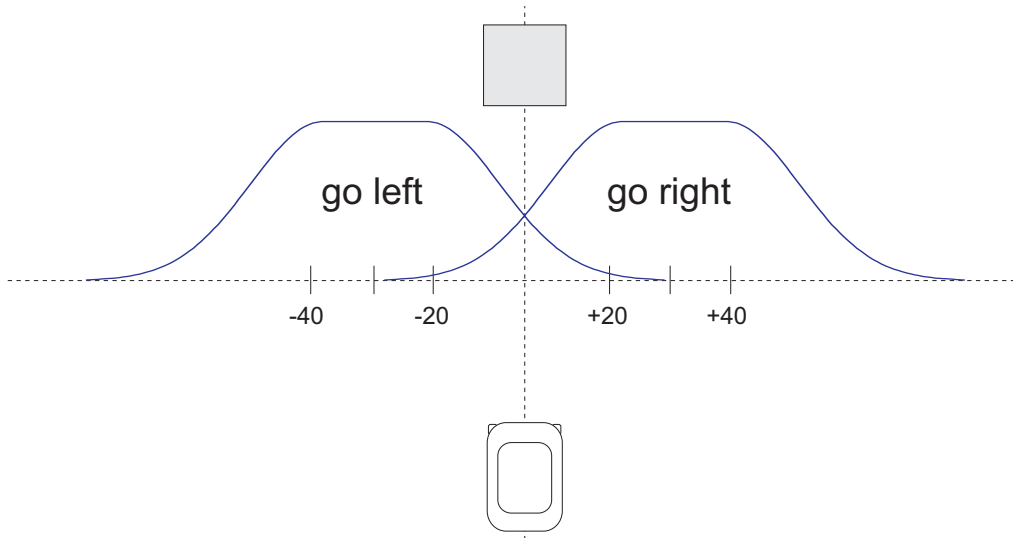


Figure 1: Car navigation example.

Suppose that the obstacle is located just at zero angle position, then both rules definitely fire. Under the conjunctive representation of the rule base, the resulting output fuzzy set before defuzzification has a camel-like shape due to its bimodal character. Indeed, take two fuzzy sets from Fig. 1 and combine them by the maximum operator (or by another fuzzy disjunction). Applying the common MOM or COG defuzzification methods the fuzzy system navigates the car just to crash into the obstacle. This is the consequence of inconsistency of the rules; irrespective of the fact that any of two rules, if applied alone, provides a good navigation policy.

In the case of the implicative representation, two fuzzy sets are combined by a fuzzy conjunction, see Section 4.2 for details, and one is interested in the core of the resulting output fuzzy set. In the above case, the core is empty which indicates inconsistency. Indeed, the left output fuzzy set admits as fully compatible navigation the angles in range $[-40^\circ, -20^\circ]$ and the other in range $[+20^\circ, +40^\circ]$. These intervals are then combined by the intersection which correspond to conjunctive combination in the computational model of the implicative fuzzy systems. Hence the implicative systems have capabilities to detect inconsistencies in their rule bases.

However, two issues arise here. One is that in today world of data abundance the rule bases are mostly created automatically by machine learning algorithms from empirical data and detection of inconsistencies is forced to be

performed on-line on the basis of characteristics of the created fuzzy systems. Note that the data-drive rule bases consists of dozens or even hundreds of rules and an expert assessment of consistency is impossible here. This makes a serious problem especially when designing adaptive fuzzy controllers [1, 18].

The second issue deals with generalization of inconsistency detection when going over the set of all possible inputs into the fuzzy system. In the above example, we have detected inconsistency for one fixed input (0° angle location of the obstacle), but what about if the obstacle was located at a different position, say -30° . Would the two rules be still inconsistent? And what about all other possible locations of the obstacle? Generally speaking, we ask if for a given rule base and for any possible input it can't happen that the rule base is inconsistent. This quantification of consistency over all possible inputs leads to the notion of coherence of the fuzzy implicative system [7].

In the paper, we show that both issues can be effectively handled for the class of the radial fuzzy systems, namely for the radial implicative fuzzy systems. This makes a difference from ordinary fuzzy systems where checking coherence leads to a somehow complicated and customized procedures driven by the shape of employed fuzzy sets [7, 17]. In the case of the radial systems, a unified view of the coherence checking procedure is developed.

The radial conjunctive fuzzy systems introduced in the paper are also highly relevant. They are the most common system used in applications and they have close relation to neural networks [10]. Duality in the representation of the radial rule bases leads to interesting synergies when conjunctive systems are syntactically transformed into the implicative ones [3].

3. Preliminaries

This section briefly reviews several notions from fuzzy set theory in order to set up the employed notation. The review draws on textbooks and handbooks on fuzzy set theory [25, 15, 6, 20, 21] and other more or less specialized literature [14, 9]. In order to distinguish the fuzzy sets from the ordinary ones, we call the latter the *crisp sets*.

Definition 1. *Let X be a crisp set called the universe. A fuzzy set A specified on the universe X is any mapping μ_A from X to the interval $[0, 1]$. This mapping is commonly called the membership function of the fuzzy set A . We use $A(x)$ instead of $\mu_A(x)$ to simplify the notation.*

Definition 2. $\mathcal{F}(X)$ stands for the set of all fuzzy sets specified on X .

The special case of universa are the Cartesian products. Fuzzy sets specified on the Cartesian products of crisp sets are called *fuzzy relations*.

In the definitions below, we assume that the fuzzy set A is specified on some universe X .

Definition 3. The fuzzy set A is called *normal* if there exists at least one element of X such that $A(x) = 1$.

Definition 4. The height of the fuzzy set A , denoted $\text{height}(A)$, is the largest membership degree to A , formally, $\text{height}(A) = \sup_{x \in X} \{A(x)\}$.

Definition 5. The support of the fuzzy set A , denoted $\text{supp}(A)$, is the crisp subset of X such that its elements have a non-zero membership degree to A , formally, $\text{supp}(A) = \{x \in X \mid A(x) > 0\}$.

Definition 6. The core of the fuzzy set A , denoted $\text{core}(A)$, is the crisp subset of the elements of X such that $A(x) = 1$, formally, $\text{core}(A) = \{x \in X \mid A(x) = 1\}$.

Definition 7. Let $\alpha \in [0, 1]$. The α -cut of the fuzzy set A , denoted $[A]^\alpha$, is the crisp subset of X specified as $[A]^\alpha = \{x \in X \mid A(x) \geq \alpha\}$.

By inspecting the above definitions, one may identify several relations such as $[A]^0 = X$ or $[A]^1 = \text{core}(A)$.

3.1. t -norms and fuzzy conjunctions

In fuzzy set theory, operations on the fuzzy sets are tightly related to the notions of the t -norm, s -norm and fuzzy implication. We recall their definitions and basic properties.

Definition 8. The t -norm T is a function from $[0, 1] \times [0, 1]$ to $[0, 1]$ satisfying the following four properties:

- (T1) $T(a, b) = T(b, a)$ (commutativity),
- (T2) $T(a, T(b, c)) = T(T(a, b), c)$ (associativity),
- (T3) if $a_1 \leq a_2$ then $T(a_1, b) \leq T(a_2, b)$ (monotonicity),
- (T4) $T(a, 1) = a$ (boundary condition).

It is well known that the t -norms extend the classical Boolean conjunction because one has $T(1, 1) = 1$ and $T(1, 0) = T(0, 1) = T(0, 0) = 0$. That is why they are called the *fuzzy conjunctions* and employed to specify the fuzzy intersections of fuzzy sets.

Definition 9. Let A_1, \dots, A_m , $m \in \mathbb{N}$ be the fuzzy sets specified on a common universum X . Their fuzzy intersection is the fuzzy set $\bigwedge_{j=1}^m A_j$ with the membership function specified on the basis of some t -norm T as

$$\left(\bigwedge_{j=1}^m A_j\right)(x) = T(A_1(x), \dots, A_m(x)). \quad (1)$$

In the following sections, we will usually denote the t -norms by the star symbol \star and use the multiplicative notation $T(a, b) = a \star b$. Using this notation the aforementioned formula writes

$$\left(\bigwedge_{j=1}^m A_j\right)(x) = A_1(x) \star \dots \star A_m(x). \quad (2)$$

Different t -norms determine different fuzzy conjunctions/intersections. The best known examples of the t -norms are the minimum t -norm $T_M(a, b) = \min\{a, b\}$, the product t -norm $T_P(a, b) = a \cdot b$ and the Łukasiewicz t -norm $T_L(a, b) = \max\{0, a + b - 1\}$ for $a, b \in [0, 1]$.

Another often imposed requirement on the t -norms is their continuity: (T5) T is a continuous function.

The *Archimedean t -norms* play an important role in the theory of t -norms [14, 15]. These t -norms possess the Archimedean property that states that for any $a, b \in (0, 1)$ there exists an $n \in \mathbb{N}$ such that (using the multiplicative notation) $T(a, \dots, a) = a \star \dots \star a < b$, and a is presented n -times in the formula.

The continuous Archimedean t -norms form an important class because of their well known characterization theorem [14]. In order to review the theorem, we recall the concept of the continuous additive (decreasing) generator of a t -norm.

Definition 10. The t -norm T has a continuous additive generator if there exists a continuous, strictly decreasing function $t : [0, 1] \rightarrow [0, +\infty]$, $t(1) = 0$,

which is uniquely determined up to a positive multiplicative constant, such that for all $a, b \in [0, 1]$

$$T(a, b) = t^{(-1)}(t(a) + t(b)) \quad (3)$$

where $t^{(-1)}$ is the pseudo-inverse of t .

The pseudo-inverse $t^{(-1)}$ of the continuous additive generator t is the function from $[0, +\infty]$ to $[0, 1]$ which is specified as

$$t^{(-1)}(z) = \begin{cases} t^{-1}(z) & \text{for } z \in [0, t(0)] \\ 0 & \text{for } z \in (t(0), +\infty] \end{cases} \quad (4)$$

where t^{-1} is the ordinary inverse of t .

Based on the above formula, one may specify the pseudo-inverse of the continuous additive generator using the ordinary inverse as

$$t^{(-1)}(z) = t^{-1}(\min\{t(0), z\}), \quad z \in [0, \infty]. \quad (5)$$

Having the concept of the continuous additive generator defined, the representation theorem for the continuous Archimedean t -norms reads as follows:

Theorem 1. *T is a continuous Archimedean t -norm if and only if T has a continuous additive generator.*

Proof. See [14], Sec. 5.1 on page 122. □

3.2. s -norms and fuzzy disjunctions

The counterparts to the t -norms are the so-called s -norms (or t -conorms). They extend the Boolean disjunction and are also called as the *fuzzy disjunctions*. A similar definition to Definition 9 is established that defines the s -norms, see [15]. Because we will not work extensively with the s -norms in this paper, we do not go into more details here. We only mention that the s -norms are employed to establish the fuzzy unions of fuzzy sets:

Definition 11. *Let A_1, \dots, A_m , $m \in \mathbb{N}$ be fuzzy sets defined on a common universum X . Their fuzzy union is the fuzzy set $\bigvee_{j=1}^m A_j$ with the membership function specified on the basis of some s -norm S as*

$$\left(\bigvee_{j=1}^m A_j\right)(x) = S(A_1(x), \dots, A_m(x)). \quad (6)$$

The most common s -norm used in applications is the maximum s -norm, i.e., $S(a, b) = \max\{a, b\}$ for $a, b \in [0, 1]$.

3.3. Residuated fuzzy implications

In fuzzy set theory, fuzzy implications form several classes. They are operations from the unit square to the unit interval and extend the classical Boolean implication. In this paper, we are interested in the so-called *residuated implications* or *R-implications* in short. The *R-implications* are derived from the *t-norms* by the operation of residuation [15, 9].

Definition 12. *Let \star be the *t-norm*. The residuated implication \rightarrow_\star is the operation on the unit square specified as*

$$a \rightarrow_\star b = \sup_z \{z \in [0, 1] \mid z \star a \leq b\}. \quad (7)$$

Based on this definition, one can easily see that for any *R-implication* it holds that $a \rightarrow_\star b = 1$ iff $a \leq b$.

The residuated implications that are derived from the minimum, product and Łukasiewicz *t-norms* write for $a, b \in [0, 1]$ as follows:

- $a \rightarrow_M b = 1$ for $a \leq b$ and $a \rightarrow_M b = b$ for $a > b$ - the so-called Gödel implication;
- $a \rightarrow_P b = 1$ for $a \leq b$ and $a \rightarrow_P b = b/a$ for $a > b$ - the so-called Goguen implication;
- $a \rightarrow_L b = 1$ for $a \leq b$ and $a \rightarrow_L b = 1 - a + b$ for $a > b$ - the so-called Łukasiewicz implication.

4. Basics of fuzzy systems

In this section, we review the basics of fuzzy systems. More profound reviews are available in [20, 25, 27, 15]. We introduce the general concept and specialize it to what is called the *conjunctive and implicative fuzzy systems in the standard configuration*. We present the computational models of both these systems.

4.1. General fuzzy system

The fuzzy system encodes a function from an input crisp set X to an output crisp set Y . A concrete computational form of this function draws on the mathematical representations of the four building blocks of the fuzzy system: a *fuzzifier*, a *rule base*, an *inference engine* and a *defuzzifier*. In the

quadruple, the second and the third blocks constitute the so-called *proper fuzzy system*. It processes a fuzzy set on its input and yields a fuzzy set on its output. The fuzzifier is just a device for mapping points of the input set X to fuzzy sets specified on X . The defuzzifier then maps fuzzy sets specified on Y into points of Y .

The rule base of the fuzzy system bears the knowledge stored in the system. Canonically, this knowledge is represented by a (crisp) set of IF-THEN rules. The IF-THEN rules represent fuzzy relations on the input-output set $X \times Y$. These individual fuzzy relations are combined into the overall fuzzy relation RB that mathematically represents the rule base.

When the fuzzy system is in use, an input into the system $x^* \in X$ is fuzzified in the fuzzifier to a fuzzy set A' specified on X . The fuzzy set A' is combined with the stored knowledge in the inference engine. As the output, the fuzzy set $B' \in \mathcal{F}(Y)$ is released. Thus, the inference engine performs a mapping from $\mathcal{F}(X) \times \mathcal{F}(X \times Y)$ to $\mathcal{F}(Y)$. Finally, the fuzzy set B' is defuzzified in the defuzzifier into a point $y^* \in Y$ that constitutes the output from the fuzzy system. This flow of computation determines the function computed by the fuzzy system. Let us now state the formal definition of the fuzzy system.

Definition 13. *The fuzzy system is the 6-tuple $\{X, Y, fuzz, RB, IE, defuzz\}$ where*

- X is an input crisp set,
- Y is an output crisp set,
- $fuzz : X \rightarrow \mathcal{F}(X)$ is a mapping from X to $\mathcal{F}(X)$ (the fuzzifier),
- $defuzz : \mathcal{F}(Y) \rightarrow Y$ is a mapping from $\mathcal{F}(Y)$ to Y (the defuzzifier),
- the rule base RB is a fuzzy relation specified on the Cartesian product $X \times Y$, i.e., $RB \in \mathcal{F}(X \times Y)$,
- the inference engine IE is a mapping from $\mathcal{F}(X) \times \mathcal{F}(X \times Y)$ to $\mathcal{F}(Y)$.

The fuzzy system determines the function $FS : X \rightarrow Y$ that is specified as

$$FS : y^* = defuzz(IE(fuzz(x^*), RB)). \quad (8)$$

The above definition associates the fuzzy system FS with a certain function. If needed, this function will be explicitly indicated by using the argument $x^* \in X$. That is, by $FS(x^*)$ we will mean the function (8) that is related to the fuzzy system FS .

In Definition 13, the rule base is specified on a general level as a fuzzy relation. In what follows, we will specialize this specification to the notion of the MISO rule base.

4.2. Conjunctive and implicative MISO rule bases

In Definition 13, we work with general input and output sets. In real applications, however, the MISO (multiple-input single-output) configuration of the fuzzy system and its rule base is the most common. Under this configuration, $X \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$ and $Y \subseteq \mathbb{R}$; and the corresponding rule bases are called the MISO rule bases.

There are two basic ways how individual IF-THEN rules are interpreted and combined into the final rule base: the *conjunctive* and the *implicative* way.

In the *conjunctive rule bases*, the antecedents (the IF parts) and consequents (the THEN parts) of the IF-THEN rules are combined by a fuzzy conjunction and individual rules by a fuzzy disjunction. The stored knowledge is viewed as the parallel list (the fuzzy disjunction) of input-output examples (fuzzy points) from the relation the rule base represents. The conjunctive rule bases are seen as driven by examples.

In the *implicative rule bases*, the antecedents are combined with the consequents by a fuzzy implication and individual rules by a fuzzy disjunction. In this case, the IF-THEN rules correspond to the set of restrictions the represented relation must satisfy simultaneously (the fuzzy conjunction). This way of the representation is seen as logically driven.

Let us be more specific. Let the rule base of the fuzzy system consist of $m \in \mathbb{N}$ IF-THEN rules. The j -th IF-THEN rule encodes the fuzzy relation R_j specified on the $X \times Y \subseteq \mathbb{R}^{n+1}$ set with the membership function

$$R_j(\mathbf{x}, y) = A_{j1}(x_1) \star \cdots \star A_{jn}(x_n) \triangleright B_j(y). \quad (9)$$

In the formula, A_{ji} , $j = 1, \dots, m$, $i = 1, \dots, n$ stand for the one-dimensional fuzzy sets implementing the antecedent of the j -th rule and B_j is the fuzzy set that implements the consequent of the rule. The \star operation corresponds to the fuzzy conjunction which is used to form the antecedent of the j -th rule

$$A_j(\mathbf{x}) = A_{j1}(x_1) \star \cdots \star A_{jn}(x_n). \quad (10)$$

The \triangleright operation corresponds either to the fuzzy conjunction \star used in (10) - for the conjunctive systems; or to the fuzzy implication \rightarrow_\star in the case of the implicative systems. In the latter case, we stress that \rightarrow_\star is the R -implication that is derived from the t -norm \star that is used to build the antecedents (10) for $j = 1, \dots, m$.

The fuzzy relation (9) can be expressed in the more compact form employing only the antecedent and consequent of the rule:

$$R_j(\mathbf{x}, y) = A_j(\mathbf{x}) \triangleright B_j(y). \quad (11)$$

In the above formulas, clearly, one has $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ and A_j is the multidimensional fuzzy set that is specified on the input set $X \subseteq \mathbb{R}^n$.

Using the notation of (11), one has the following two formulas to represent the conjunctive and implicative MISO rule bases:

$$RB_{conj}(\mathbf{x}, y) = \bigvee_{j=1}^m \{A_j(\mathbf{x}) \star B_j(y)\}, \quad RB_{impl}(\mathbf{x}, y) = \bigwedge_{j=1}^m \{A_j(\mathbf{x}) \rightarrow B_j(y)\}.$$

In applications, the maximum s -norm is usually taken to interpret the fuzzy disjunction \bigvee ; and the minimum t -norm to interpret the fuzzy intersection \bigwedge . This leads to the following formal definition.

Definition 14. *The conjunctive or the implicative MISO rule base is the 8-tuple $RB_\triangleright = \{X, Y, n, m, \star, \{\{A_{ji}\}_{i=1}^n\}_{j=1}^m, \{B_j\}_{j=1}^m, \triangleright\}$ where*

- $X \subseteq \mathbb{R}^n$ is an input crisp set,
- $Y \subseteq \mathbb{R}$ is an output crisp set,
- $n \in \mathbb{N}$ is the dimension of X ,
- $m \in \mathbb{N}$ is the number of IF-THEN rules constituting the rule base,
- \star is a t -norm,
- $\{\{A_{ji}\}_{i=1}^n\}_{j=1}^m$ is the set of $n \cdot m$ one-dimensional fuzzy sets forming the antecedents of IF-THEN rules where $A_j = A_{j1} \star \dots \star A_{jn}$, $j = 1, \dots, m$, i.e., $A_j \in \mathcal{F}(X \subseteq \mathbb{R}^n)$, $j = 1, \dots, m$,
- $\{B_j\}_{j=1}^m$ is the set of m fuzzy sets forming the consequents of IF-THEN rules, i.e., $B_j \in \mathcal{F}(Y \subseteq \mathbb{R})$, $j = 1, \dots, m$,

- \triangleright corresponds either to the t -norm \star , i.e., $\triangleright = \star$, for the conjunctive rule base; or to the residuated fuzzy implication derived from \star , i.e., $\triangleright = \rightarrow_\star$, for the implicative rule base, respectively.

The specifications of the corresponding fuzzy relations write as

$$RB_{conj}(\mathbf{x}, y) = \max_{j=1}^m \{A_j(\mathbf{x}) \star B_j(y)\} \quad (12)$$

for the conjunctive rule base; and

$$RB_{impl}(\mathbf{x}, y) = \min_{j=1}^m \{A_j(\mathbf{x}) \rightarrow B_j(y)\} \quad (13)$$

for the implicative rule base.

By inspecting the specification of the \triangleright operation in the above definition, we see that in the conjunctive rule bases, an individual rule is build up using the identical t -norm \star . That is, the identical \star is used to build up antecedents and to combine them with the consequents. In the implicative rule bases, the operation of residuation (7) gives the relation between the \star of the antecedent (10) and the employed residuated fuzzy implication \rightarrow_\star .

4.3. Singleton fuzzifier and CRI engine

We proceed our specialization by recalling the *singleton fuzzifier* and the *CRI engine* that are the most common choices for the fuzzifier and the inference engine in applications.

Definition 15. Let $x^* \in X$ be the input to the fuzzy system. The singleton fuzzifier relates its input to the fuzzy set A' with the membership function

$$A'(x) = \begin{cases} 1 & \text{for } x = x^* \\ 0 & \text{otherwise.} \end{cases} \quad (14)$$

The CRI engine implements the well known *compositional rule of inference* [25, 15] for composing a fuzzy set with a fuzzy relation. The definition of the engine reads as follows:

Definition 16. Let A be a fuzzy set specified on X and RB a fuzzy relation specified on $X \times Y$. The CRI engine specifies the membership function of the output fuzzy set B' , $B' \in \mathcal{F}(Y)$, as

$$B'(y) = \sup_{x \in X} \{A(x) \star RB_{\triangleright}(x, y)\} \quad (15)$$

for $A \in \mathcal{F}(X)$, $RB \in \mathcal{F}(X \times Y)$ and \star being some t -norm.

The popularity of the combination of the singleton fuzzifier and the CRI engine stems from the fact that when A' is specified according to (14), then (15) writes as

$$B'(y) = RB(x^*, y). \quad (16)$$

This is due to the properties of the t -norms, namely that $T(0, a) = 0$ and $T(1, a) = a$ for any t -norm T and $a \in [0, 1]$. Clearly, this choice greatly simplifies the computation of the CRI engine; and it does not depend on the explicit form of \star in (15). However, there are other inference engines available. As an example, let us mention the engine that is based on the Bandler-Kohout subproduct [24].

4.4. WA and MOM methods of defuzzification

Discussing methods of defuzzification, there is plenty of them proposed in literature [25, 15, 4]. Here we mention the two that are suitable for the MISO systems where $Y \subseteq \mathbb{R}$. The first is the *weighted average method* (the WA method) which is the simplified version of the COG defuzzification method. The method has several other names in literature such as the center average defuzzifier [25] or the method of heights [4]. We later relate the WA method to the conjunctive fuzzy systems. The second method of defuzzification, which we later relate to the implicative fuzzy systems, is the *mean-of-maxima method* (the MOM method).

The center of gravity (COG) method of defuzzification states the defuzzified value y_B^* of the $B \in \mathcal{F}(Y)$ fuzzy set as $y_B^* = \int_Y y \cdot B(y) dy / \int_Y B(y) dy$. The point y_B^* is known as the *centroid* of the B fuzzy set. The COG method is generally computationally intensive and it might be hard to state y_B^* explicitly.

To ease the computation, one may take the advantage of the singleton fuzzifier. In this case and for the conjunctive systems, one has

$$RB_{conj}(x^*, y) = \max_{j=1}^m \{A_j(x^*) \star B_j(y)\} = \max_{j=1}^m \{B'_j(y)\}$$

where we denoted $B'_j(y) = A_j(x^*) \star B_j(y)$. A simplification to the COG method is done by considering the weighted average of the centroids of the consequent fuzzy sets B_j . These centroids can be computed in advance and do not change with the input into the fuzzy system because the input affects only the antecedents of rules $A_j(x^*)$. The weights in the average are determined as the heights of the B'_j fuzzy sets. If the consequent fuzzy sets B_j are normal, one has $\text{height}(B'_j) = A_j(x^*)$ for each $j = 1, \dots, m$. Following

this line of thinking carries out the formal definition of the WA method of defuzzification.

Definition 17. Let $Y \subseteq \mathbb{R}$ and $B(y) = \max_{j=1}^m \{B'_j(y)\}$ where $B'_j(y) = A_j(x^*) \star B_j(y)$ for a given input $x^* \in X$. Let there exist at least one $j \in \{1, \dots, m\}$ such that $A_j(x^*) > 0$. Let the centroids of B_j sets, $y_{B_j}^* = \int y \cdot B_j(y) dy / \int B_j(y) dy$, exist and $y_{B_j}^* \in \mathbb{R}$ for all $j = 1, \dots, m$. Then the weighted average method of defuzzification is specified as

$$y^* = \frac{\sum_{j=1}^m A_j(x^*) \cdot y_{B_j}^*}{\sum_{j=1}^m A_j(x^*)}. \quad (17)$$

By inspecting the above formula, we see that it is defined only when the centroids $y_{B_j}^*$ exist and for those inputs for which there exists at least one rule j that is firing ($A_j(x^*) > 0$); because otherwise we would have the denominator equal to zero in (17).

The denominator is zero if none of the rules is firing, i.e., $A_j(x^*) = 0$ for all $j = 1, \dots, m$. What should be the output of the fuzzy system in this situation? The technical solution could be considering the limit of (17) for the denominator approaching zero, but it does not exist. However, considering the problem methodologically, we should not allow that this situation happens, because if it happened, then it would mean that the knowledge stored in the fuzzy system is irrelevant to such the input. Adopting the idea that the stored knowledge must be relevant to all possible inputs leads to the requirement that for every $x^* \in X$ there exists at least one rule j such that $A_j(x^*) > 0$. Such rule bases are called complete [25, 4].

Definition 18. Degree of covering of a rule base is specified as

$$\text{DOC} = \inf_{x \in X} \{\max_{j=1}^m \{A_j(x)\}\}. \quad (18)$$

The rule base of a fuzzy system is called complete if $\text{DOC} > 0$.

Note that the notions of DOC and completeness are relevant to the general rule bases, not only to the MISO ones. Now, we are going to switch to the MOM method of defuzzification.

Definition 19. Let $Y \subseteq \mathbb{R}$ and $B \in \mathcal{F}(Y)$ be normal. Let both $\inf\{\text{core}(B)\}$ and $\sup\{\text{core}(B)\}$ be finite. Then the mean-of-maxima defuzzifier is specified as

$$y^* = \frac{\inf\{\text{core}(B)\} + \sup\{\text{core}(B)\}}{2}. \quad (19)$$

In the above definition, we assume that both infimum and supremum of $\text{core}(B)$ are finite. Later we will see that in the case of the implicative fuzzy systems, the $\text{core}(B)$ is specified as the intersection of the cores of the fuzzy sets that are derived from the consequents B_j ; and the requirement on finiteness can be assured by a reasonable choice of the B_j sets. For example, we may require that they form the fuzzy numbers [14, 15, 25].

4.5. Conjunctive and implicative fuzzy systems in the standard configuration

We end the section by formally introducing the *conjunctive and implicative fuzzy system in the standard configuration*. We call them in the standard configuration because this configuration prevails in the real-world applications.

Definition 20. *Let FS be the fuzzy system. We call it the conjunctive fuzzy system in the standard configuration if $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}$. FS has the conjunctive MISO rule base, uses the CRI inference engine and employs the singleton fuzzifier and the WA method of defuzzification.*

Definition 21. *Let FS be the fuzzy system. We call it the implicative fuzzy system in the standard configuration if $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}$. FS has the implicative MISO rule base, uses the CRI inference engine and employs the singleton fuzzifier and the MOM method of defuzzification.*

In what follows, we are interested in the computational models of both types of these fuzzy systems, i.e., in the explicit forms of the associated $FS : X \subseteq \mathbb{R}^n \rightarrow Y \subseteq \mathbb{R}$ functions.

Lemma 1. *Let FS_{conj} be the conjunctive fuzzy system in the standard configuration. Let the rule base of the fuzzy system be complete and the centroids $y_{B_j}^* \in \mathbb{R}$ of the B_j sets exists for all $j = 1, \dots, m$. Then the fuzzy system computes the function*

$$FS_{conj}(\mathbf{x}^*) = \frac{\sum_{j=1}^m A_j(\mathbf{x}^*) \cdot y_{B_j}^*}{\sum_{j=1}^m A_j(\mathbf{x}^*)}. \quad (20)$$

Proof. If the singleton fuzzifier is employed, then the output of the CRI inference engine writes $B'(y) = RB(\mathbf{x}^*, y) = \max_{j=1}^m \{A_j(\mathbf{x}^*) \star B_j(y)\}$. The

application of the WA method of defuzzification (17) gives the output (20) under two assumptions. The first one is the completeness of the rule base; and the other is the existence and finiteness of the centroids of the B_j sets. \square

When we switch to the implicative fuzzy systems, we will find as crucial the notion of *coherence of the implicative fuzzy system* [7, 4]. In words, coherence assures that for each possible input there is specified a fully reliable output from the implicative fuzzy system. Mathematically, we require that the output fuzzy set from the inference engine is always normal. This is equivalent to the requirement that the core of the inference engine's output fuzzy set is non-empty, as the emptiness would mean that for the given input we have contradictory rules in the rule base.

Definition 22. *The implicative fuzzy system in the standard configuration is **coherent** if for any input $\mathbf{x}^* \in X$ the fuzzy set B' that is released from the inference engine is normal.*

The following two lemmas state equivalent conditions for coherence and bring a certain insight into the computational model of the implicative fuzzy systems in the standard configuration.

Lemma 2. *For any input $\mathbf{x}^* \in X$ into the implicative fuzzy system in the standard configuration, the core of the B' fuzzy set writes as*

$$\text{core}(B') = \bigcap_{j=1}^m [B_j]^{A_j(\mathbf{x}^*)} \quad (21)$$

where on the right-hand side there is the intersection of the crisp sets $[B_j]^{A_j(\mathbf{x}^*)}$ which are the $A_j(\mathbf{x}^*)$ -cuts of the consequents fuzzy sets B_j .

Proof. Because the fuzzy system is in the standard configuration, one has $B'(y) = \min_{j=1}^m \{A_j(\mathbf{x}^*) \rightarrow B_j(y)\}$. Furthermore, we have $[B_j]^{A_j(\mathbf{x}^*)} = \{y \mid A_j(\mathbf{x}^*) \leq B_j(y)\}$ by Definition 7 of the α -cut.

1) Let us show that $\text{core}(B') \subseteq \bigcap_{j=1}^m [B_j]^{A_j(\mathbf{x}^*)}$. If $\text{core}(B') = \emptyset$, then it trivially holds. Let $y \in \text{core}(B')$, i.e., $B'(y) = 1$, then due to the specification of the B' set, $B'(y) = \min_j \{A_j(\mathbf{x}^*) \rightarrow B_j(y)\}$, and the properties of the t -norms ($T(a_1, \dots, a_n) = a_1 \star \dots \star a_n = 1$ iff $a_j = 1$ for each j), it must be $B_j'(y) = 1$ for every j as well. For the residuated implications one has $I(a, b) = 1$ iff $a \leq b$, thus the inequality $A_j(\mathbf{x}^*) \leq B_j(y)$ holds for

every j . In other words, $y \in [B_j]^{A_j(\mathbf{x}^*)}$ for all $j = 1, \dots, m$ and therefore $y \in \bigcap_{j=1}^m [B_j]^{A_j(\mathbf{x}^*)}$.

2) Let us show that $\bigcap_{j=1}^m [B_j]^{A_j(\mathbf{x}^*)} \subseteq \text{core}(B')$. If $\bigcap_{j=1}^m [B_j]^{A_j(\mathbf{x}^*)} = \emptyset$, then it trivially holds. Let $y \in \bigcap_{j=1}^m [B_j]^{A_j(\mathbf{x}^*)}$, then $A_j(\mathbf{x}^*) \leq B_j(y)$ holds for every $j = 1, \dots, m$. So one has also $\min_{j=1}^m \{A_j(\mathbf{x}^*) \rightarrow B(y)\} = 1$ and therefore $y \in \text{core}(B')$. \square

Theorem 2. *The implicative fuzzy system in the standard configuration is coherent if and only if for any $\mathbf{x}^* \in X$ it holds that*

$$\bigcap_{j=1}^m [B_j]^{A_j(\mathbf{x}^*)} \neq \emptyset. \quad (22)$$

Proof. Obviously, a fuzzy set is normal if and only if its core is non-empty. From the form of the core of B' in the implicative fuzzy systems in the standard configuration we have the statement of the theorem. \square

Having the notion of coherence defined, we may approach to the specification of the computational model of the implicative fuzzy systems in the standard configuration.

Lemma 3. *Let FS_{impl} be the coherent implicative fuzzy system in the standard configuration and both $\inf\{\text{core}(B')\}$ and $\sup\{\text{core}(B')\}$ be finite for any input $\mathbf{x}^* \in X$. Then the fuzzy system computes the function*

$$FS_{impl}(\mathbf{x}^*) = \frac{\inf\{\bigcap_{j=1}^m [B_j]^{A_j(\mathbf{x}^*)}\} + \sup\{\bigcap_{j=1}^m [B_j]^{A_j(\mathbf{x}^*)}\}}{2}. \quad (23)$$

Proof. Clearly, the formula (23) is the result of when the assertion of Lemma 2 is applied to the formula (19). Note that we have B' set always normal due to the assumed coherence of the system. \square

From the above lemmas, we see that in order to have the concrete computational models effectively specified one has to meet several assumptions. In the case of the conjunctive fuzzy systems, these are the assumptions of the completeness of the rule base and specification of the finite centroids. In the case of the implicative fuzzy systems, the coherence is crucial because one wants to have consistent rules in the rule base. Further, both minima and suprema of $\text{core}(B')$ must be finite. In what follows we will show that these assumptions can be treated in a rather convenient way in the class of the radial fuzzy systems.

5. Radial fuzzy systems

Radial fuzzy systems form a subclass of the fuzzy systems in the standard configuration. Hence, we have $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}$ in these systems. The fuzzy sets employed within the radial fuzzy systems are implemented using radial functions. Moreover, in the radial fuzzy systems, antecedents of the IF-THEN rules are required to exhibit a certain shape preservation property.

The specification of the radial fuzzy systems draws on the specification of the radial rule bases. Before we state the associated definition, we recall the concepts of the *radial function* and the *scaled ℓ_p norm*.

5.1. Radial functions and scaled ℓ_p norms

In words, the radial function is a function that is invariant to changes in its argument that do not affect the argument's distance to a central point of the function. A prototypical example of such a change is rotation. Mathematically, this property is encoded by incorporating a norm of a difference between the argument and the selected central point. The distance of the argument from the central point is further modified by another univariate function.

Definition 23. A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, $n \in \mathbb{N}$ is called *radial* if it has the form $f(\mathbf{x}) = \Phi(\|\mathbf{x} - \mathbf{a}\|)$ where $\|\cdot\|$ is a norm in \mathbb{R}^n , Φ is a function from $[0, +\infty)$ to \mathbb{R} and $\mathbf{a} \in \mathbb{R}^n$ is the central point of the radial function f .

In the above definition, the function Φ is sometimes called the *shape function* and is usually non-negative and monotonic.

Concerning norms in \mathbb{R}^n , the best known class is the class of the ℓ_p norms. We extend this class by introducing the class of the scaled ℓ_p norms.

Definition 24. Let $\mathbf{b} = (b_1, \dots, b_n)$, $b_i > 0$, $i = 1, \dots, n$, $n \in \mathbb{N}$. The *scaled ℓ_p norm* $\|\cdot\|_{p,\mathbf{b}}$ is the norm specified for $p \in [1, \infty]$ as follows:

$$\begin{aligned} \|\mathbf{u}\|_{p,\mathbf{b}} &= (|u_1/b_1|^p + \dots + |u_n/b_n|^p)^{1/p} \quad \text{for } p \in [1, +\infty), \\ \|\mathbf{u}\|_{\infty,\mathbf{b}} &= \lim_{p \rightarrow \infty} \|\mathbf{u}\|_{p,\mathbf{b}} = \max_i \{|u_i/b_i|\}. \end{aligned} \tag{24}$$

As for the standard ℓ_p norms, the settings $p = 1$, $p = 2$ and $p = \infty$ provide the most frequently used selections. The corresponding norms are the scaled ℓ_1 (or octaedric), scaled ℓ_2 (or Euclidean) and scaled ℓ_∞ (or cubic norm), respectively. In the first and the third case, the names are derived from the shapes of the unit ball which is the octaeder or cube in \mathbb{R}^3 space, respectively.

5.2. Radial rule bases

Here we put forward the notion of the *radial rule base* that is subsequently used to introduce the radial fuzzy systems.

Definition 25. The MISO rule base $RB_{\triangleright} = \{X, Y, n, m, \star, \{\{A_{ji}\}_{i=1}^n\}_{j=1}^m, \{B_j\}_{j=1}^m, \triangleright\}$ is called *radial* if the following three conditions are satisfied:

- (i) There exists a continuous function $act : [0, +\infty) \rightarrow [0, 1]$, $act(0) = 1$ such that (a) either there exists $z_0 \in (0, +\infty)$ such that act is strictly decreasing on $[0, z_0]$ and $act(z) = 0$ for $z \in [z_0, +\infty)$ or (b) act is strictly decreasing on $[0, +\infty)$ and $\lim_{z \rightarrow +\infty} act(z) = 0$.
- (ii) Fuzzy sets in the antecedent and consequent of the j -th rule are specified as

$$A_{ji}(x_i) = act\left(\frac{|x_i - a_{ji}|}{b_{ji}}\right), \quad (25)$$

$$B_j(y) = act\left(\frac{\max\{0, |y - c_j| - s_j\}}{d_j}\right) \quad (26)$$

where $n, m \in \mathbb{N}$; $i = 1, \dots, n$; $j = 1, \dots, m$; $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} = (x_1, \dots, x_n)$; $\mathbf{a}_j \in \mathbb{R}^n$, $\mathbf{a}_j = (a_{j1}, \dots, a_{jn})$; $\mathbf{b}_j \in \mathbb{R}_+^n$, $\mathbf{b}_j = (b_{j1}, \dots, b_{jn})$, $b_{ji} > 0$; $c_j \in \mathbb{R}$; $d_j \in \mathbb{R}$, $d_j > 0$; $s_j \in \mathbb{R}$, $s_j \geq 0$.

- (iii) For each $\mathbf{x} \in X \subseteq \mathbb{R}^n$ the radial property holds, i.e.,

$$A_j(\mathbf{x}) = A_{j1}(x_1) \star \dots \star A_{jn}(x_n) = act(\|\mathbf{x} - \mathbf{a}_j\|_{p, \mathbf{b}_j}), \quad (27)$$

where $\|\cdot\|_{p, \mathbf{b}_j}$ is the scaled version of an ℓ_p norm $\|\cdot\|_p$. This ℓ_p norm is common to all rules in the rule base.

Generally speaking, in the radial fuzzy systems, the IF-THEN rules have specific form that makes their computational model simpler and mathematically tractable, especially in the case of the implicative systems as we will see later.

In the definition, there are presented three conditions that make a MISO rule base to be radial. The first two conditions (i) and (ii) deal with the membership functions of the fuzzy sets forming the antecedents and consequents of the IF-THEN rules. Since the act function is required to be a non-increasing continuous function from $[0, +\infty)$ to $[0, 1]$ such that $act(0) = 1$, $\lim_{z \rightarrow +\infty} act(z) = 0$, see Fig. 2, the particular fuzzy sets form the fuzzy numbers [14, 15, 25].

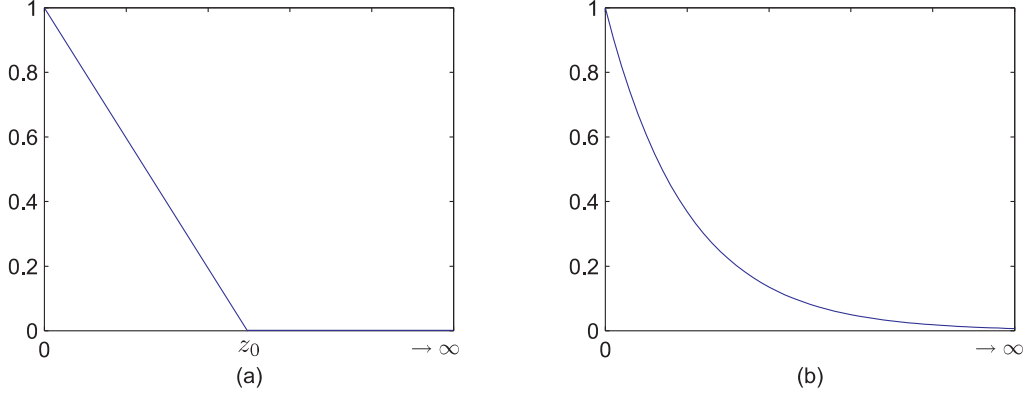


Figure 2: Examples of the act function.

In Definition 23, considering $n = 1$, the ℓ_1 norm (the absolute value) and setting $\Phi(z) = act(z/b_{ji})$ or $\Phi(z) = act(\max\{0, z - s_j\}/d_j)$ for $z \in [0, +\infty)$ we get the specifications of the one-dimensional fuzzy sets in condition (ii). Hence, in the radial rule bases, the membership functions of one-dimensional fuzzy sets are the radial functions.

The difference between the specification of the antecedents and consequents lies in the fact that the antecedents correspond to the generalized triangular fuzzy numbers and the consequents to the generalized trapezoidal fuzzy numbers. See graphical examples in Section 5.5.

The third condition (iii), which is called the *radial property* and expressed by the equality (27), is crucial for the specification of the radial fuzzy systems.

5.3. Radial property

The radial property says that the combination of the one-dimensional fuzzy sets preserves the radial shape. That is, by functional equation (27) it is required that the combination of the one-dimensional fuzzy sets A_{ji} by the employed t -norm \star must form a multidimensional fuzzy set determined by the application of the act function (the same as it is used for the A_{ji} sets) on the scaled ℓ_p norm $\|\cdot\|_{p, \mathbf{b}_j}$ of the distance of the argument \mathbf{x} from the central point \mathbf{a}_j . Moreover, it is required that the scaling vector \mathbf{b}_j is composed from the parameters b_{ji} , $i = 1, \dots, n$ of the individual one-dimensional fuzzy sets, and the central point \mathbf{a}_j from the respective central points a_{ji} , $i = 1, \dots, n$.

The radial property (27) is non-trivial because the selection of the t -norm \star and specification of the A_{ji} sets already determines the multidimensional membership function of the antecedent A_j .

Example. *Non-triviality of the radial property.* In this example, we will show that the combination of the triangular fuzzy sets by product does not exhibit the radial property.

Let the antecedent of the IF-THEN rule be formed by two triangular fuzzy sets:

$$A_1(x_1) = \max\{0, 1 - |x_1 - a_1|/b_1\}, \quad A_2(x_2) = \max\{0, 1 - |x_2 - a_2|/b_2\};$$

and let the employed t -norm \star be the product t -norm. Obviously, both sets satisfy conditions (i) and (ii) of Definition 25 and the act function corresponds to $act(z) = \max\{0, 1 - z\}$, $z \in [0, +\infty)$. Using the product t -norm, the membership function of the antecedent writes

$$A(\mathbf{x}) = \max\{0, 1 - |x_1 - a_1|/b_1\} \cdot \max\{0, 1 - |x_2 - a_2|/b_2\}.$$

Let us show that there is no ℓ_p norm such that (27) holds for every $\mathbf{x} \in \mathbb{R}^2$.

Denoting $u_1 = (x_1 - a_1)/b_1$, $u_2 = (x_2 - a_2)/b_2$, the radial property (27) reads as

$$\max\{0, 1 - |u_1|\} \cdot \max\{0, 1 - |u_2|\} = \max\{0, 1 - \|(u_1, u_2)\|_p\}. \quad (28)$$

Let $\mathbf{x} = (x_1, x_2)$ be set in such a way that $u_1 = u_2 = u \in (0, 1)$ and assume that (28) holds. We get

$$(1 - |u|)^2 = 1 - \|(u, u)\|_p. \quad (29)$$

Note that the right-hand side cannot be zero because the left-hand side is positive for our choice of $u \in (0, 1)$. The equality (29) makes the value of the norm $\|(u, u)\|_p$ to be $\|(u, u)\|_p = 1 - (1 - |u|)^2$; and therefore

$$\|(u, u)\|_p = 2|u| - |u|^2. \quad (30)$$

It is clear that (30) cannot specify a norm because for any $u \in (0, 1)$ and $\lambda \in (0, 1)$ we obtain

$$\|(\lambda u, \lambda u)\|_p = 2|\lambda u| - |\lambda u|^2 = |\lambda| \cdot (2|u| - |u|^2), \quad (31)$$

which contradicts the homogeneity property of the norms that requires that $\|c\mathbf{u}\| = |c| \cdot \|\mathbf{u}\|$ for any $c \in \mathbb{R}$.

By this example we see that the radial property is non-trivial and we naturally ask which the t -norms and shapes (the act functions) can be combined so that the radial property holds? This is the topic of the next section.

To end this section, remark that if the radial property holds, then the antecedents A_j are represented by the multivariate radial functions. Indeed, if the radial property holds, then we have $A_j(\mathbf{x}) = act(\|\mathbf{x} - \mathbf{a}_j\|_p, b_j)$, which fits the specification of the radial function given in Definition 23. A link to the radial basis functions of RBF neural networks theory can be established here [10].

Moreover, the radial property brings other advantages. Let us mention the following two. First, the computational model of the related radial fuzzy systems, especially of the implicative ones, is mathematically tractable. Second, when creating a rule base from data, the clustering techniques are commonly used to do the job, e.g., the fuzzy c -means algorithm is of popular use in this area [11]. The identified clusters then correspond to the antecedents and consequents of the IF-THEN rules and they can be straightforwardly transformed into the membership functions of the radial fuzzy sets.

5.4. Building radial rule bases

Having the concept of the radial rule base specified, there is a natural request for examples of such the rule bases, especially with respect to the some previously selected t -norms. Let us start with the minimum t -norm.

Theorem 3. *Let the MISO rule base with m rules be specified as follows:*

- 1) *The employed t -norm is the minimum t -norm.*
- 2) *The act function is any function from $[0, +\infty)$ to $[0, 1]$ meeting the condition (i) of Definition 25.*
- 3) *The fuzzy sets forming the antecedents and consequents have the form*

$$A_{ji}(x_i) = act\left(\frac{|x_i - a_{ji}|}{b_{ji}}\right), \quad B_j(y) = act\left(\frac{\max\{0, |y - c_j| - s_j\}}{d_j}\right) \quad (32)$$

for $j = 1, \dots, m$; $\mathbf{a}_j \in \mathbb{R}^n$; $\mathbf{b}_j \in \mathbb{R}_+^n$; $c_j \in \mathbb{R}$; $d_j \in \mathbb{R}$, $d_j > 0$; $s_j \in \mathbb{R}$, $s_j \geq 0$.

Then the rule base is radial.

Proof. The conditions 2) and 3) of the specification coincide with the conditions (i) and (ii) of Definition 25. Hence it remains to prove that using the minimum t -norm validates the radial property (27).

As the *act* function is non-increasing we have

$$\min\{act(u_1), \dots, act(u_n)\} = act(\max\{u_1, \dots, u_n\})$$

for $u_i \in [0, +\infty)$, $i = 1, \dots, n$. Therefore, if the conditions 2) and 3) are satisfied and we use the minimum t -norm we obtain the antecedents in form

$$A_j(\mathbf{x}) = \min_i \left\{ act \left(\frac{|x_i - a_{ji}|}{b_{ji}} \right) \right\} = act \left(\max_i \left\{ \frac{|x_i - a_{ji}|}{b_{ji}} \right\} \right).$$

Because $b_{ji} > 0$ for every j and i , we can use the scaled cubic norm to rewrite the above equality as

$$A_j(\mathbf{x}) = \min_i \left\{ act \left(\frac{|x_i - a_{ji}|}{b_{ji}} \right) \right\} = act(\|\mathbf{x} - \mathbf{a}_j\|_{\infty, \mathbf{b}_j}),$$

which is exactly the radial property (iii) of Definition 25. \square

We see that for the minimum t -norm we can assure the validity of the radial property by a suitable, but non-restrictive choice of the *act* function. Now, we aim at the continuous Archimedean t -norms.

Theorem 4. *Let the MISO rule base with m rules be specified as follows:*

- 1) *The employed t -norm T is continuous Archimedean with the continuous additive generator t and its pseudo-inverse $t^{(-1)}$.*
- 2) *For parameters $q > 0$ and $p \in [1, +\infty)$ the *act* function has the form*

$$act(z) = t^{(-1)}(qz^p), \quad z \in [0, +\infty). \quad (33)$$

- 3) *The fuzzy sets forming the antecedents and consequents are specified as*

$$A_{ji}(x_i) = t^{(-1)} \left[q \left(\frac{|x_i - a_{ji}|}{b_{ji}} \right)^p \right], B_j(y) = t^{(-1)} \left[q \left(\frac{\max\{0, |y - c_j| - s_j\}}{d_j} \right)^p \right]. \quad (34)$$

for $j = 1, \dots, m$; $\mathbf{a}_j \in \mathbb{R}^n$; $\mathbf{b}_j \in \mathbb{R}_+^n$; $c_j \in \mathbb{R}$; $d_j \in \mathbb{R}$, $d_j > 0$; $s_j \in \mathbb{R}$, $s_j \geq 0$.

Then the rule base is radial.

Proof. Having the *act* function given by (33), the specification of the fuzzy sets according to (34) coincides with the formulas (25) and (26). What we need to show is that the radial property (27) holds and the specification of the *act* function meets the condition (i) of Definition 25.

Using the additive generator, we can write T as $T(a, b) = t^{(-1)}(t(a) + t(b))$. Let us show that we have also $T(a, T(b, c)) = t^{(-1)}(t(a) + t(b) + t(c))$.

In order to do that, we recall from Section 3 that the pseudo-inverse of the continuous additive generator t of an Archimedean t -norm is defined as

$$t^{(-1)}(z) = \begin{cases} t^{-1}(z) & \text{for } z \in [0, t(0)], \\ 0 & \text{for } z \in (t(0), +\infty] \end{cases} \quad (35)$$

and therefore the following formulas hold:

$$t^{(-1)}(t(z)) = z \quad \text{for } z \in [0, 1], \quad (36)$$

$$t(t^{(-1)}(z)) = \begin{cases} z & \text{for } z \in [0, t(0)], \\ t(0) & \text{for } z \in (t(0), +\infty]. \end{cases} \quad (37)$$

We have $T(a, T(b, c)) = t^{(-1)}(t(a) + t(t^{(-1)}(t(b) + t(c))))$ and there are two cases possible:

- 1) $t(b) + t(c) \leq t(0)$. According to (37), $t(t^{(-1)}(t(b) + t(c))) = t(b) + t(c)$; and therefore $T(a, T(b, c)) = t^{(-1)}(t(a) + t(b) + t(c))$.
- 2) $t(b) + t(c) > t(0)$. By (37), $t(t^{(-1)}(t(b) + t(c))) = t(0)$; and therefore $T(a, T(b, c)) = t^{(-1)}(t(a) + t(0))$. Since $t(a) + t(0) \geq t(0)$, we obtain from (35) $T(a, T(b, c)) = 0$. But $t^{(-1)}(t(a) + t(b) + t(c)) = 0$ as $t(a) + t(b) + t(c) > t(0)$, hence $T(a, T(b, c)) = t^{(-1)}(t(a) + t(b) + t(c))$ as well.

By induction and the same approach it can be shown that for any $n \in \mathbb{N}$, $n > 1$ and $u_i \in [0, 1]$, $i = 1, \dots, n$ it also holds

$$T(u_1, \dots, u_n) = t^{(-1)}(t(u_1) + \dots + t(u_n)). \quad (38)$$

Now, let the act and A_{ji} functions be specified according to (33) and (34). We denote $u_{ji} = (x_i - a_{ji})/b_{ji}$ for $i = 1, \dots, n$, $\mathbf{a}_j = (a_{j1}, \dots, a_{jn})$, $\mathbf{b}_j = (b_{j1}, \dots, b_{jn})$. Then, using (38), the representation of the antecedents $A_j(\mathbf{x})$ writes as

$$A_j(\mathbf{x}) = t^{(-1)} \left[\sum_{i=1}^n t(t^{(-1)}(q|u_{ji}|^p)) \right]. \quad (39)$$

By (37), one reads the above as

$$A_j(\mathbf{x}) = t^{(-1)} \left[\sum_{i=1}^n \min\{t(0), q|u_{ji}|^p\} \right]. \quad (40)$$

We have again two cases possible: 1) If $q|u_{ji}|^p < t(0)$ for all i , then (40) has the form $A_j(\mathbf{x}) = t^{(-1)}(\sum_{i=1}^n q|u_{ji}|^p)$, which can be clearly written as $A_j(\mathbf{x}) = t^{(-1)}(q[\sqrt[p]{\sum_{i=1}^n |u_{ji}|^p}]^p)$, i.e., $A_j(\mathbf{x}) = act(\|\mathbf{u}\|_p) = act(\|\mathbf{x} - \mathbf{a}_j\|_{p, \mathbf{b}_j})$.

2) If there exists an i such that $q|u_{ji}|^p \geq t(0)$, then, on the one hand, the sum in (40) is greater or equal to $t(0)$ and therefore $A_j(\mathbf{x}) = 0$. On the other hand, $\sum_{i=1}^n q|u_{ji}|^p \geq t(0)$, i.e., $q[\sqrt[p]{\sum_{i=1}^n |u_{ji}|^p}]^p = q\|\mathbf{x} - \mathbf{a}_j\|_{p, \mathbf{b}_j}^p \geq t(0)$ and therefore $act(\|\mathbf{x} - \mathbf{a}_j\|_{p, \mathbf{b}_j}) = 0$. Thus, $A_j(\mathbf{x}) = act(\|\mathbf{x} - \mathbf{a}_j\|_{p, \mathbf{b}_j})$ also in this case and we see that the specification of the *act* function according to (33) is sufficient for the validity of the radial property.

In order to complete the proof, we must show that the function $t^{(-1)}(qz^p)$, $q > 0$, $p \in [1, +\infty)$, $z \in [0, +\infty)$ can be considered as the *act* function. For $q = 1$, $p = 1$ consider the properties of the continuous additive generator and its pseudo-inverse:

1) Either the value of $t(0)$ is bounded, i.e., $t(0) < +\infty$. Then $t^{(-1)}$ is continuous, strictly decreasing on $[0, t(0)]$ and equals to zero on $[t(0), +\infty]$. Thus, $t^{(-1)}$ is continuous and non-increasing on the interval $[0, +\infty]$.

2) Or the value of $t(0)$ is unbounded, i.e., $t(0) = +\infty$. Then $t^{(-1)}$ is continuous, strictly decreasing on $[0, +\infty]$ and $t^{(-1)}(t(0)) = t^{(-1)}(+\infty) = 0$, which means that $\lim_{z \rightarrow +\infty} t^{(-1)}(z) = 0$.

Hence $t^{(-1)}(z)$ meets the conditions of Definition 25 on the *act* function.

Since the function qz^p , $z \in [0, +\infty)$ is a continuous strictly increasing bijection on $[0, +\infty)$ for $q > 0$, $p \in [1, +\infty)$, the function $t^{(-1)}(qz^p)$ can be considered as the *act* function too. \square

The proved theorems give us tools for constructing the radial rule bases and subsequently the *radial fuzzy systems*. Several examples of these systems are presented below.

5.5. Radial fuzzy systems

Here we deliver the formal definitions of the radial fuzzy systems. We will distinguish between the conjunctive and implicative radial fuzzy systems.

Definition 26. Let *FS* be the fuzzy system. We call it **the radial conjunctive fuzzy system** (radial C-FS) if it is the conjunctive fuzzy system in the standard configuration and its rule base is radial.

Definition 27. Let *FS* be the fuzzy system. We call it **the radial implicative fuzzy system** (radial I-FS) if it is the implicative fuzzy system in the standard configuration and its rule base is radial.

By inspecting the definitions, we see that the fuzzy system is radial if it is in the standard configuration and has the radial rule base. The definitions enable us to identify computational models of the radial fuzzy systems and to study their properties. Before we discuss these computational models, we present two concrete examples of the radial fuzzy systems. The first system is based on the minimum t -norm and the other on the product t -norm.

5.5.1. Mamdani radial conjunctive fuzzy system

The name of this radial fuzzy system refers to the pioneering work of Mamdani and Assilian who used the minimum t -norm to build a fuzzy controller [19].

The Mamdani radial C-FS use the minimum t -norm $T_M(a, b) = \min\{a, b\}$ and the triangular act function: $act(z) = \max\{0, 1 - z\}$, $z \in [0, +\infty)$ which leads to the following membership functions of the respective fuzzy sets:

$$A_{ji}^M(x_i) = \max\left\{0, 1 - \frac{|x_i - a_{ji}|}{b_{ji}}\right\},$$

$$B_j^M(y) = \max\left\{0, 1 - \frac{\max\{0, |y - c_j| - s_j\}}{d_j}\right\}$$

where $\mathbf{a}_j \in \mathbb{R}^n$, $\mathbf{a}_j = (a_{j1}, \dots, a_{jn})$; $\mathbf{b}_j \in \mathbb{R}_+^n$, $\mathbf{b}_j = (b_{j1}, \dots, b_{jn})$, $b_{ji} > 0$; $c_j \in \mathbb{R}$; $d_j \in \mathbb{R}$, $d_j > 0$; $s_j \in \mathbb{R}$, $s_j \geq 0$ are parameters. Examples of these membership functions are presented in Fig. 3.

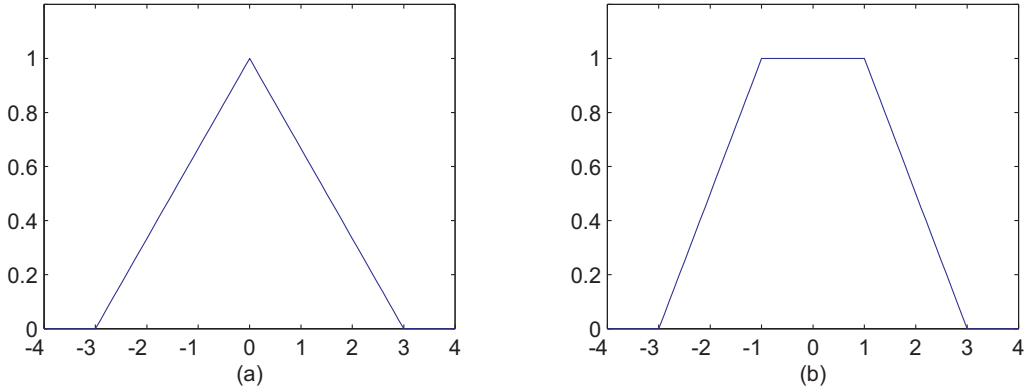


Figure 3: The Mamdani radial C-FS. (a) An example of the antecedent fuzzy set for $a = 0$, $b = 3$; (b) An example of the consequent fuzzy set for $c = 0$, $d = 2$, $s = 1$.

Clearly, the *act* function $act(z) = \max\{0, 1 - z\}$, $z \in [0, +\infty)$ does satisfy the requirement (ia) of Definition 25 for $z_0 = 1$. Moreover, since the minimum t -norm is used, the assumptions of Theorem 3 are fulfilled and the Mamdani C-FS is really radial, i.e., it exhibits the radial property.

Due to the radial property, the antecedent of the j -th rule in the Mamdani system writes

$$A_j^M(\mathbf{x}) = \min\{A_{j1}^M(x_1), \dots, A_{jn}^M(x_n)\} = \max\{0, 1 - \|\mathbf{x} - \mathbf{a}_j\|_{\infty, \mathbf{b}_j}\}.$$

Thus, the common ℓ_p norm is the cubic norm. Fig. 4 presents examples of the antecedent and the rule base in the Mamdani radial C-FS.

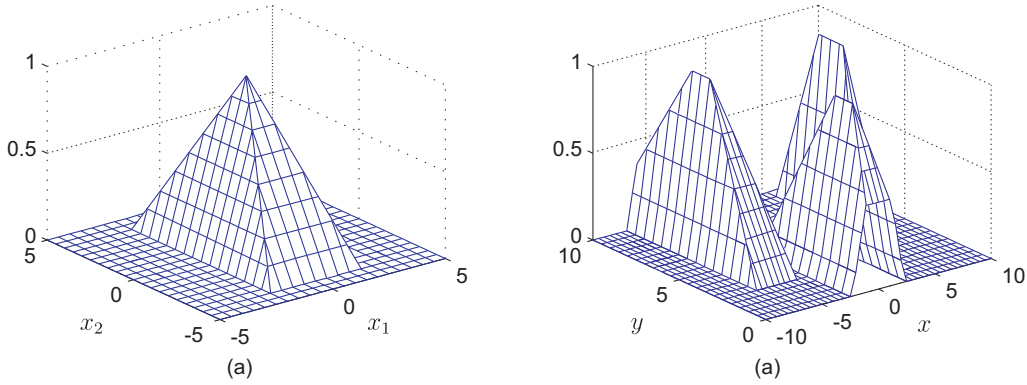


Figure 4: The Mamdani radial C-FS. (a) An example of the antecedent for $n = 2$, $\mathbf{a} = (0, 0)$, $\mathbf{b} = (2, 4)$; (b) An example of the rule base for $n = 2$, $m = 3$.

5.5.2. Gaussian radial implicative fuzzy system

This radial fuzzy system is considered as the implicative one. The system uses the product t -norm $T_P(a, b) = a \cdot b$ and the corresponding residuated implication which is the Goguen implication. Recall that this implication is specified as $a \rightarrow_P b = 1$ for $a \leq b$ and $a \rightarrow_P b = b/a$ for $a > b$, $a, b \in [0, 1]$.

The product t -norm is continuous Archimedean. The pseudo-inverse of its additive generator corresponds to the exponential function: $t^{(-1)}(z) = \exp(-z)$ for $z \in [0, +\infty)$, $\exp(-\infty) = 0$. Theorem 4 is employed to obtain the specification of the fuzzy sets in the Gaussian radial I-FS. Using the above pseudo-inverse $t^{(-1)}$, setting $q = 1$ and choosing $p = 2$, the membership

functions of the respective fuzzy sets writes as

$$A_{ji}^G(x_i) = \exp \left[-\frac{(x_i - a_{ji})^2}{b_{ji}^2} \right], \quad (41)$$

$$B_j^G(y) = \exp \left[-\frac{\max\{0, |y - c_j| - s_j\}^2}{d_j^2} \right]. \quad (42)$$

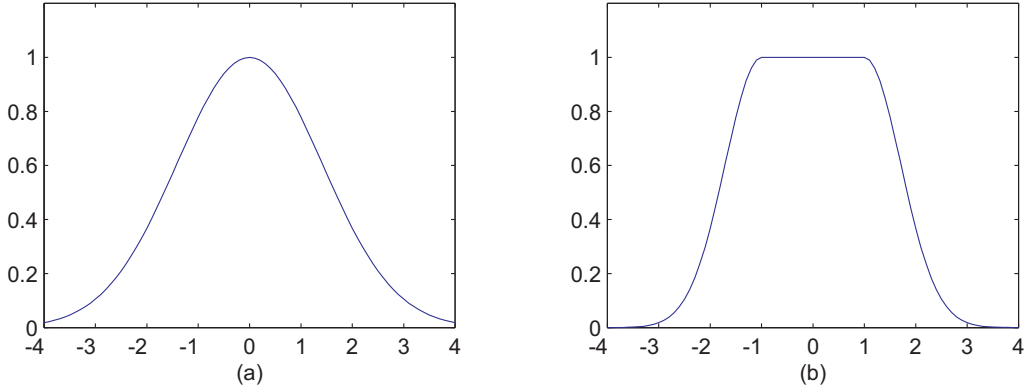


Figure 5: The Gaussian radial I-FS. (a) An example of the antecedent fuzzy set for $a = 0$, $b = 2$; (b) An example of the consequent fuzzy set for $c = 0$, $d = 1$, $s = 1$.

The *act* function in the Gaussian radial I-FS system is determined by the exponential function in the form $act(z) = \exp(-z^2)$, $z \in [0, \infty)$. Hence the requirement (ib) of Definition 25 applies and the assumptions of Theorem 4 are fulfilled.

We can see that the membership functions of the antecedent fuzzy sets $A_{ji}^G(x)$ coincide with the Gaussian curves. The membership functions of the consequent fuzzy sets $B_j^G(y)$ are then given by their trapezoidal modification. Fig. 5 presents examples of both types of fuzzy sets.

By simple computation, the antecedent of the j -th rule in the Gaussian I-FS writes as

$$A_j^G(\mathbf{x}) = A_{j1}^G(x_1) \cdot \dots \cdot A_{jn}^G(x_n) = \exp(-\|\mathbf{x} - \mathbf{a}_j\|_{2, \mathbf{b}_j}). \quad (43)$$

Thus, the common ℓ_p norm is the Euclidean norm.

In Fig. 6(a) an example of the antecedent is presented for $n = 2$. In Fig. 6(b) there is presented the fuzzy relation that corresponds to the rule base of the Gaussian radial I-FS that consists of three rules.

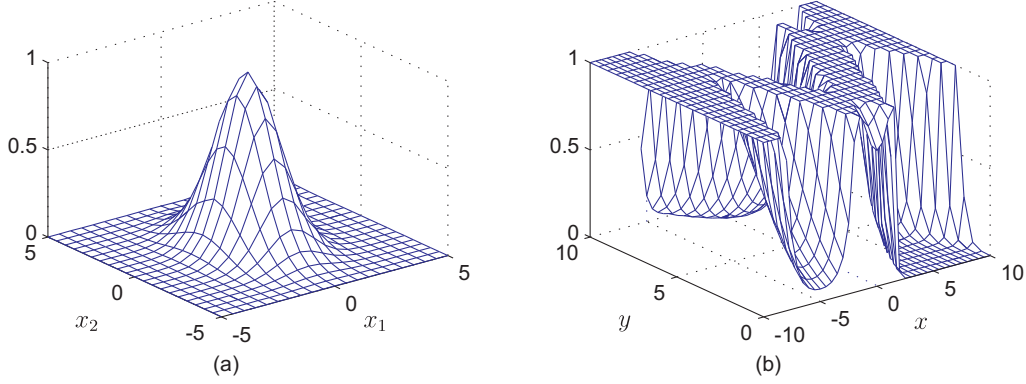


Figure 6: The Gaussian radial I-FS. (a) An example of the antecedent for $n = 2$, $\mathbf{a} = (0, 0)$, $\mathbf{b} = (2, 4)$; (b) An example of the rule base for $n = 1$, $m = 3$.

6. Computational models of radial fuzzy systems

In this section we discuss the computational models of the radial conjunctive and implicative fuzzy systems. We show that the radial property makes the latter mathematically tractable. We also investigate the mutual relationship between both models. To proceed, let us stress that from now on we will use bare \mathbf{x} to denote the input into the fuzzy system instead of the former \mathbf{x}^* .

6.1. Computational model of radial conjunctive fuzzy systems

The situation is rather straightforward here. Due to the radial property and the radial character of the consequent fuzzy sets we have the following modification of Lemma 1 for the radial conjunctive fuzzy systems:

Lemma 4. *Let FS_{rconj} be the radial conjunctive fuzzy system. Let its rule base be complete. Then the fuzzy system computes the function*

$$FS_{rconj}(\mathbf{x}) = \frac{\sum_{j=1}^m act(\|\mathbf{x} - \mathbf{a}_j\|_{p, \mathbf{b}_j}) \cdot c_j}{\sum_{j=1}^m act(\|\mathbf{x} - \mathbf{a}_j\|_{p, \mathbf{b}_j})}. \quad (44)$$

Proof. To adapt the formula (20) of Lemma 1, we put forward two observations. First, the antecedents write $A_j(\mathbf{x}) = act(\|\mathbf{x} - \mathbf{a}_j\|_{p, \mathbf{b}_j})$ due to the radial property. Second, the centroids of the consequents fuzzy sets B_j correspond to their central points c_j because of the radial symmetry of the B_j sets. \square

6.2. Computational model of radial implicative fuzzy systems

In Section 4.5, we have shown that the computational model of the implicative fuzzy systems draws on the specification of the α -cuts

$$[B_j]^{A_j(\mathbf{x})} = \{y \mid A_j(\mathbf{x}) \leq B_j(y)\}. \quad (45)$$

Let us show that these α -cuts can be stated explicitly in the case of the radial implicative fuzzy systems.

Definition 28. *For an input $\mathbf{x} \in X$, the output of the j -th rule of the radial implicative fuzzy system is the interval*

$$I_j(\mathbf{x}) = \begin{cases} (-\infty, +\infty) & \text{for } A_j(\mathbf{x}) = 0 \\ I_j^+(\mathbf{x}) & \text{for } A_j(\mathbf{x}) > 0 \end{cases} \quad (46)$$

where the interval

$$I_j^+(\mathbf{x}) = [c_j - d_j \|\mathbf{x} - \mathbf{a}_j\|_{p, \mathbf{b}_j} - s_j, c_j + d_j \|\mathbf{x} - \mathbf{a}_j\|_{p, \mathbf{b}_j} + s_j] \quad (47)$$

is called the positive part of the output of the j -th rule for the input $\mathbf{x} \in X$.

Theorem 5. *In the radial I-FS, for any $\mathbf{x} \in X$ and $j = 1, \dots, m$, the $[B_j]^{A_j(\mathbf{x})}$ coincides with the output of the j -th rule, i.e.,*

$$[B_j]^{A_j(\mathbf{x})} = I_j(\mathbf{x}). \quad (48)$$

Proof. For a given $\mathbf{x} \in X$, if $A_j(\mathbf{x}) = 0$, then $[B_j]^0 = (-\infty, +\infty)$ by Definition 7 of the α -cut.

If $A_j(\mathbf{x}) > 0$, then we have the respective α -cut specified by the formula (45), which can be expanded into the following chain of the equivalent inequalities:

$$\begin{aligned} A_j(\mathbf{x}) &\leq B_j(y), \\ \text{act}(\|\mathbf{x} - \mathbf{a}_j\|_{p, \mathbf{b}_j}) &\leq \text{act}(\max\{0, |y - c_j| - s_j\}/d_j), \\ \|\mathbf{x} - \mathbf{a}_j\|_{p, \mathbf{b}_j} &\geq \max\{0, |y - c_j| - s_j\}/d_j, \\ d_j \|\mathbf{x} - \mathbf{a}_j\|_{p, \mathbf{b}_j} &\geq \max\{0, |y - c_j| - s_j\}, \\ d_j \|\mathbf{x} - \mathbf{a}_j\|_{p, \mathbf{b}_j} &\geq |y - c_j| - s_j, \\ d_j \|\mathbf{x} - \mathbf{a}_j\|_{p, \mathbf{b}_j} + s_j &\geq |y - c_j|. \end{aligned} \quad (49)$$

In the chain, only the step from the second to the third inequality should be explained. By Definition 25, the *act* function is generally non-increasing

and strictly decreasing on interval $[0, z_0)$ for some $z_0 \in (0, \infty]$. Further, we have $act(z_+) > 0$ for $z_+ \in [0, z_0)$. Hence, if $0 < act(z_+) \leq act(z)$, then it must be that $z_+ \geq z$. As we have assumed that $0 < act(\|\mathbf{x} - \mathbf{a}_j\|_p, \mathbf{b}_j)$, the discussed step is correct. \square

The inequality (49) determines the closed interval which can be considered as the output of the j -th rule for the input $\mathbf{x} \in X$ if $A_j(\mathbf{x}) > 0$. It is this fact that is reflected in Definition 28 when labeling the interval $I_j^+(\mathbf{x})$. Note that the limit points of $I_j^+(\mathbf{x})$ depend on all parameters $\mathbf{a}_j, \mathbf{b}_j, c_j, d_j$ and s_j .

To proceed, we prove a simple lemma dealing with the intersection of closed intervals in \mathbb{R} .

Lemma 5. *Let $\{I_j\}_{j=1}^m$ be a set of m non-empty closed intervals in \mathbb{R} . Let $L(I_j)$ or $R(I_j)$ denotes the left or the right limit point of the j -th interval, respectively, i.e., $I_j = [L(I_j), R(I_j)]$. Then the intersection $\bigcap_{j=1}^m I_j$ is non-empty if and only if*

$$\max_j \{L(I_j)\} \leq \min_j \{R(I_j)\}; \quad (50)$$

and we have

$$\bigcap_{j=1}^m I_j = [\max_j \{L(I_j)\}, \min_j \{R(I_j)\}]. \quad (51)$$

Proof. Because we work with the closed intervals in \mathbb{R} , each interval writes as $I_j = [L(I_j), +\infty) \cap (-\infty, R(I_j)]$. The intersection of m intervals is then given as

$$\bigcap_{j=1}^m I_j = [L(I_1), +\infty) \cap (-\infty, R(I_1)] \cap \dots \cap [L(I_m), +\infty) \cap (-\infty, R(I_m)].$$

Due to the commutativity and associativity of the intersection, the right-hand side has the form

$$([L(I_1), +\infty) \cap \dots \cap [L(I_m), +\infty)] \cap \dots \cap (-\infty, R(I_1)] \cap \dots \cap (-\infty, R(I_m)])$$

and therefore $\bigcap_{j=1}^m I_j = [\max_j \{L(I_j)\}, +\infty) \cap (-\infty, \min_j \{R(I_j)\}]$. This gives a non-empty set if and only if $\max_j \{L(I_j)\} \leq \min_j \{R(I_j)\}$; and in this case

$$\bigcap_{j=1}^m I_j = [\max_j \{L(I_j)\}, \min_j \{R(I_j)\}].$$

This finishes the proof. \square

Lemma 6. *Let FS_{rimpl} be the coherent radial implicative fuzzy system. Let its rule base be complete. Then the fuzzy system computes the function*

$$FS_{rimpl}(\mathbf{x}) = \frac{\max_{j|A_j(\mathbf{x})>0}\{L(I_j^+(\mathbf{x}))\} + \min_{j|A_j(\mathbf{x})>0}\{R(I_j^+(\mathbf{x}))\}}{2}. \quad (52)$$

Proof. As we assume that the rule base of the fuzzy system is complete, the set of indices $\{j|A_j(\mathbf{x}) > 0\}$ is non-empty for any input $\mathbf{x} \in X$. Further, because of completeness and coherence we have also

$$\bigcap_{j|A_j(\mathbf{x})>0} I_j^+(\mathbf{x}) = \bigcap_{j=1}^m I_j(\mathbf{x}) \neq \emptyset$$

for any input $\mathbf{x} \in X$. Definition 28 and Lemma 5 give us the specification of the left-hand side of the above equality, i.e.,

$$\bigcap_{j|A_j(\mathbf{x})>0} I_j^+(\mathbf{x}) = [\max_{j|A_j(\mathbf{x})>0}\{L(I_j^+(\mathbf{x}))\}, \min_{j|A_j(\mathbf{x})>0}\{R(I_j^+(\mathbf{x}))\}].$$

According to Theorem 5, the right-hand side corresponds to the intersection of the $[B_j]^{A_j(\mathbf{x})}$ α -cuts, which are the cores of the fuzzy sets issued from the inference engine, and the assertion is obtained by Lemma 3. \square

6.3. Relation between computational models

The following lemma tells us that under the assumptions of completeness and coherence, the computations of the above models cannot differ arbitrarily much.

Lemma 7. *Let RB be the complete radial rule base. Let FS_{rconj} and FS_{rimpl} be the radial fuzzy systems based on this RB . Let FS_{rimpl} be coherent, then*

$$|FS_{rconj}(\mathbf{x}) - FS_{rimpl}(\mathbf{x})| < (c_{max} - c_{min})$$

for every $\mathbf{x} \in X$ where $c_{max} = \max_{j=1}^m \{c_j\}$ and $c_{min} = \min_{j=1}^m \{c_j\}$.

Proof. c_j are the centroids of the consequent fuzzy sets B_j , $j = 1, \dots, m$. First, we have $FS_{rconj}(\mathbf{x}) \in [c_{min}, c_{max}]$, because the output of FS_{rconj} is given as a weighted average of c_j ; and it is well known that the weighted average lies in-between the minima and maxima of the weighted points.

We further argue that also $FS_{rimpl}(\mathbf{x}) \in [c_{min}, c_{max}]$ for any $\mathbf{x} \in X$. Let $J^+ = \{j | A_j(\mathbf{x}) > 0\}$ and $I = \bigcap_{j \in J^+} I_j^+(\mathbf{x}) = [y_{min}, y_{max}] \neq \emptyset$. Either $I \subseteq [c_{min}, c_{max}]$ and we are done or there exists a point from I that lies outside the interval $[c_{min}, c_{max}]$. Let $y_{max} > c_{max}$, we have $y_{max} \in I_j^+$ for all $j \in J^+$ because I is given by the intersection of the I_j^+ intervals. Now, let $r_{max} = y_{max} - c_{max} > 0$. We have also $c_{max} - r_{max} \in I$. Indeed, let $r_{j,max} = y_{max} - c_j$, then $c_j + r_{j,max} = y_{max} \in I_j^+$, $j \in J^+$. As I_j^+ intervals are symmetric around the central points c_j , one has also $c_j - r_{j,max} \in I_j^+$, $j \in J^+$. Further $r_{j,max} \geq r_{max}$ for each $j \in J^+$, so also $c_j - r_{max} \in I_j^+$ for each $j \in J^+$. Because $c_j \leq c_{max} < y_{max}$, and $y_{max} \in I_j^+$, we have also $c_j - r_{max} \leq c_{max} - r_{max} < y_{max} - r_{max} = c_{max} < y_{max}$ for all $j \in J^+$; and therefore $c_{max} - r_{max} \in I$.

Because $c_{max} - r_{max} \in I$, we have $y_{min} \leq c_{max} - r_{max}$, so $y_{min} + y_{max} \leq c_{max} - r_{max} + r_{max} + c_{max}$, i.e., $y^* = (y_{min} + y_{max})/2 \leq c_{max}$. Now, if $c_{min} \leq y_{min}$ then we are done. If $y_{min} < c_{min}$, then we get in the same way as above that $c_{min} + r_{min} \in I$ for $r_{min} = c_{min} - y_{min}$. This gives $y_{max} \geq c_{min} + r_{min}$, $y^* = (y_{min} + y_{max})/2 \geq c_{min}$ and we finally have $y^* \in [c_{min}, c_{max}]$. \square

7. Coherence of radial implicative fuzzy systems

In this section, we deal with the coherence of the radial implicative fuzzy systems. The main result presented here is the specification of a sufficient condition for a radial I-FS to be coherent. This sufficient condition depends only on the parameters of the system. We start by two straightforward lemmas.

Lemma 8. *The radial I-FS is coherent if and only if for any input $\mathbf{x} \in X$ the intersection of the outputs of its rules is non-empty, i.e., if for every $\mathbf{x} \in X$*

$$\bigcap_{j=1}^m I_j(\mathbf{x}) \neq \emptyset. \quad (53)$$

Proof. The assertion is a direct corollary of Theorems 2 and 5. \square

Since we always have $I_j^+(\mathbf{x}) \subseteq I_j(\mathbf{x})$, no matter what the value of $A_j(\mathbf{x})$ is, we can formulate the following more convenient sufficient condition for the radial I-FS to be coherent.

Lemma 9. *In the radial I-FS, if the intersection of the positive parts of the outputs of rules is non-empty for any input $\mathbf{x} \in X$, i.e., if*

$$\bigcap_{j=1}^m I_j^+(\mathbf{x}) \neq \emptyset \quad (54)$$

for every $\mathbf{x} \in X$, then the fuzzy system is coherent.

Proof. If $A_j(\mathbf{x}) = 0$ for a given \mathbf{x} and j , then $I_j^+(\mathbf{x}) \subset I_j(\mathbf{x}) = (-\infty, +\infty)$. If $A_j(\mathbf{x}) > 0$, then $I_j^+(\mathbf{x}) = I_j(\mathbf{x})$. Hence for any $\mathbf{x} \in X$ and $j \in \{1, \dots, m\}$ one has $I_j^+(\mathbf{x}) \subseteq I_j(\mathbf{x})$ and therefore

$$\bigcap_{j=1}^m I_j^+(\mathbf{x}) \subseteq \bigcap_{j=1}^m I_j(\mathbf{x}).$$

If $\bigcap_{j=1}^m I_j^+(\mathbf{x}) \neq \emptyset$, then also $\bigcap_{j=1}^m I_j(\mathbf{x}) \neq \emptyset$ for $\mathbf{x} \in X$ and we get the result by Lemma 8. \square

By Lemma 9, we have transformed the coherence question into the testing the non-emptiness of the intersection of closed intervals, which is more specific than testing the intersection of more general α -cuts. But the theorem is still weak in the sense that the intersection has to be checked for any input $\mathbf{x} \in X \subseteq \mathbb{R}^n$. In the rest of the section, we transform the problem further into testing only relationships among the parameters of the fuzzy system.

To simplify the notation a bit, we will write $\|\cdot\|_{\mathbf{b}_j}$ instead of $\|\cdot\|_{p, \mathbf{b}_j}$ and $\|\cdot\|$ instead of $\|\cdot\|_p$, i.e., we omit the p index keeping in mind that we are working with the ℓ_p norms.

Theorem 6. *Let the radial implicative fuzzy system consists of m rules. If for any pair of rules $j, k \in \{1, \dots, m\}$*

$$|c_j - c_k| - (s_j + s_k) \leq \min\{d_j \alpha_j, d_k \alpha_k\} \cdot \|\mathbf{a}_j - \mathbf{a}_k\| \quad (55)$$

where $\alpha_j = 1/\max_i\{b_{ji}\}$, $\alpha_k = 1/\max_i\{b_{ki}\}$, then the radial I-FS is coherent.

Proof. By Definition 28, the positive part of the output of the j -th rule is specified for an input $\mathbf{x} \in X$ as

$$I_j^+(\mathbf{x}) = [c_j - d_j \|\mathbf{x} - \mathbf{a}_j\|_{\mathbf{b}_j} - s_j, c_j + d_j \|\mathbf{x} - \mathbf{a}_j\|_{\mathbf{b}_j} + s_j]. \quad (56)$$

Having m rules in the rule base, we are given m intervals $I_j^+(\mathbf{x})$, $\mathbf{x} \in X$. We write them as $I_j^+(\mathbf{x}) = [L(I_j^+(\mathbf{x})), R(I_j^+(\mathbf{x}))]$ where

$$L(I_j^+(\mathbf{x})) = c_j - d_j \|\mathbf{x} - \mathbf{a}_j\|_{\mathbf{b}_j} - s_j \quad \text{and} \quad R(I_j^+(\mathbf{x})) = c_j + d_j \|\mathbf{x} - \mathbf{a}_j\|_{\mathbf{b}_j} + s_j.$$

According to Lemma 9, a sufficient condition for the radial I-FS to be coherent is that $\bigcap_{j=1}^m I_j^+(\mathbf{x}) \neq \emptyset$ for every $\mathbf{x} \in X$. Hence to prove our theorem it is sufficient to show that “if (55) holds for all pairs of rules $j, k \in \{1, \dots, m\}$, then (54) holds for every $\mathbf{x} \in X$ too”. This is equivalent to “if (54) does not hold for some $\mathbf{x}^* \in X$, then also (55) does not hold for some pair j, k ”. This is what we are going to prove.

Let (54) do not hold. Then, by Lemma 5, there exists $j, k \in \{1, \dots, m\}$, $j \neq k$ such that $L(I_j^+(\mathbf{x}^*)) > R(I_k^+(\mathbf{x}^*))$ for some $\mathbf{x}^* \in X$, i.e.,

$$c_j - d_j \|\mathbf{x}^* - \mathbf{a}_j\|_{\mathbf{b}_j} - s_j > c_k + d_k \|\mathbf{x}^* - \mathbf{a}_k\|_{\mathbf{b}_k} + s_k,$$

which is

$$c_j - c_k > d_j \|\mathbf{x}^* - \mathbf{a}_j\|_{\mathbf{b}_j} + d_k \|\mathbf{x}^* - \mathbf{a}_k\|_{\mathbf{b}_k} + s_j + s_k.$$

The last inequality is possible only if $(c_j - c_k) > 0$, because the right-hand side is always non-negative. Hence

$$|c_j - c_k| > d_j \|\mathbf{x}^* - \mathbf{a}_j\|_{\mathbf{b}_j} + d_k \|\mathbf{x}^* - \mathbf{a}_k\|_{\mathbf{b}_k} + s_j + s_k. \quad (57)$$

It is straightforward to observe that for the given scaled ℓ_p norm $\|\cdot\|_{\mathbf{b}}$ and its unscaled version $\|\cdot\|$ there exists a positive number α such that $\alpha \|\mathbf{u}\| \leq \|\mathbf{u}\|_{\mathbf{b}}$ for any $\mathbf{u} \in \mathbb{R}^n$. Namely, $\alpha = 1/\max\{b_i\}$. Setting $\alpha_j = 1/\max_i\{b_{ji}\}$, $\alpha_k = 1/\max_i\{b_{ki}\}$, we have $\alpha_j \|\mathbf{u}\| \leq \|\mathbf{u}\|_{\mathbf{b}_j}$, $\alpha_k \|\mathbf{u}\| \leq \|\mathbf{u}\|_{\mathbf{b}_k}$ for every $\mathbf{u} \in \mathbb{R}^n$. Denoting

$$\begin{aligned} J_{\mathbf{b}_{jk}}(\mathbf{x}) &= d_j \|\mathbf{x} - \mathbf{a}_j\|_{\mathbf{b}_j} + d_k \|\mathbf{x} - \mathbf{a}_k\|_{\mathbf{b}_k}, \\ J_{jk}(\mathbf{x}) &= d_j \alpha_j \|\mathbf{x} - \mathbf{a}_j\| + d_k \alpha_k \|\mathbf{x} - \mathbf{a}_k\| \end{aligned} \quad (58)$$

we get $J_{\mathbf{b}_{jk}}(\mathbf{x}) \geq J_{jk}(\mathbf{x})$ for any $\mathbf{x} \in \mathbb{R}^n$.

Let us search for the minimum of the $J_{jk}(\mathbf{x})$ with respect to $\mathbf{x} \in \mathbb{R}^n$. By the triangle inequality for $\|\cdot\|$, one has $\|\mathbf{x} - \mathbf{a}_j\| + \|\mathbf{x} - \mathbf{a}_k\| \geq \|\mathbf{a}_j - \mathbf{a}_k\|$ and therefore the minimum of the left-hand side is reached either for $\mathbf{x} = \mathbf{a}_j$ or $\mathbf{x} = \mathbf{a}_k$, i.e., $\min_{\mathbf{x} \in \mathbb{R}^n} \{\|\mathbf{x} - \mathbf{a}_j\| + \|\mathbf{x} - \mathbf{a}_k\|\} = \|\mathbf{a}_j - \mathbf{a}_k\|$.

Concerning the minimum of $J_{jk}(\mathbf{x})$, there are two cases possible:

1) $d_j\alpha_j \leq d_k\alpha_k$. Then

$$\begin{aligned} J_{jk}(\mathbf{x}) &= d_j\alpha_j \cdot (\|\mathbf{x} - \mathbf{a}_j\| + \|\mathbf{x} - \mathbf{a}_k\|) + (d_k\alpha_k - d_j\alpha_j) \cdot \|\mathbf{x} - \mathbf{a}_k\|, \\ \min_{\mathbf{x} \in \mathbb{R}^n} \{J_{jk}(\mathbf{x})\} &= d_j\alpha_j \|\mathbf{a}_k - \mathbf{a}_j\|. \end{aligned}$$

2) $d_j\alpha_j > d_k\alpha_k$. Then

$$\begin{aligned} J_{jk}(\mathbf{x}) &= d_k\alpha_k \cdot (\|\mathbf{x} - \mathbf{a}_j\| + \|\mathbf{x} - \mathbf{a}_k\|) + (d_j\alpha_j - d_k\alpha_k) \cdot \|\mathbf{x} - \mathbf{a}_j\|, \\ \min_{\mathbf{x} \in \mathbb{R}^n} \{J_{jk}(\mathbf{x})\} &= d_k\alpha_k \|\mathbf{a}_j - \mathbf{a}_k\|. \end{aligned}$$

Combining both cases we obtain

$$\min_{\mathbf{x} \in \mathbb{R}^n} \{J_{jk}(\mathbf{x})\} = \min\{d_j\alpha_j, d_k\alpha_k\} \cdot \|\mathbf{a}_j - \mathbf{a}_k\|. \quad (59)$$

The inequality (57) holds for some $\mathbf{x}^* \in X$, hence it must hold also for \mathbf{x} at which the right-hand side reaches its minimum over \mathbb{R}^n . Thus using (58), it follows from (57) that

$$|c_j - c_k| > \min_{\mathbf{x} \in \mathbb{R}^n} \{J_{b_{jk}}(\mathbf{x})\} + s_j + s_k. \quad (60)$$

Since $J_{b_{jk}}(\mathbf{x}) \geq J_{jk}(\mathbf{x})$, we have $|c_j - c_k| > \min_{\mathbf{x} \in \mathbb{R}^n} \{J_{jk}(\mathbf{x})\} + s_j + s_k$ as well, which by (59) gives

$$|c_j - c_k| > \min\{d_j\alpha_j, d_k\alpha_k\} \cdot \|\mathbf{a}_j - \mathbf{a}_k\| + s_j + s_k.$$

Rearranging the above we get

$$|c_j - c_k| - (s_j + s_k) > \min\{d_j\alpha_j, d_k\alpha_k\} \cdot \|\mathbf{a}_j - \mathbf{a}_k\| \quad (61)$$

which was to prove. \square

Let us comment on the theorem. What we have proved is that we are able to assure the coherence of the radial I-FS only by testing relations among values of its parameters. This substantially improves Lemma 9 because we have excluded the dependency of the coherence condition on the values of the inputs to the radial I-FS.

By analyzing the formula (61), we see that if the $\min\{d_j\alpha_j, d_k\alpha_k\}$ is fixed, then the inequality is validated by increasing $|c_j - c_k|$ or decreasing $\|\mathbf{a}_j - \mathbf{a}_k\|$.

We interpret this fact that if the radial I-FS is not coherent, then there exists a pair of rules such that the centers of the antecedents are close, but the centers of their consequents are distant. This corresponds to the intuitive meaning of the incoherence that rules having similar preconditions state contrary conclusions [4].

In order to state the coherence according to Theorem 6, $(m^2 - m)/2$ inequalities (55) have to be verified. Indeed, the full number of the inequalities is m^2 , but they are symmetric with respect to j, k and the inequality trivially holds for $j = k$.

In order to finish the section, we will compare our results with those of Dubois, Prade and Ughetto presented in [7]. Section IV of their paper, which concerns coherence of a set of parallel gradual rules [5], is relevant to our work.

In [7], the authors present three necessary and sufficient conditions for an I-FS to be coherent: Propositions 4.1, 4.4 and 4.10. Proposition 4.1 methodologically corresponds to our Theorem 2, and is practically inapplicable because it requires checking infinitely many conditions in order to state the coherence of the I-FS.

Proposition 4.4 says that, if the I-FS consists of two rules and its consequent fuzzy sets are convex, then one can form for each rule two functions of the input \mathbf{x} that delimit lower and upper bounds of the cores of the relations that correspond to the individual rules. Denoting these functions L_1, R_1 or L_2, R_2 for the first or the second rule, respectively, the I-FS is coherent if and only if $R_1(\mathbf{x}) \geq L_2(\mathbf{x})$ and $R_2(\mathbf{x}) \geq L_1(\mathbf{x})$ for every input $\mathbf{x} \in \text{supp}(A_1) \cap \text{supp}(A_2)$. This statement is rather general because an explicit specification of the bounds may be hard and their comparison must be done for every input $\mathbf{x} \in \text{supp}(A_1) \cap \text{supp}(A_2)$. Special cases simplify the situation and several algorithms performing this checking are presented in the paper, but only for the pairs of SISO (single-input single-output) rules that employ the trapezoidal fuzzy sets.

Concerning coherence of the MISO rules, in Proposition 4.8 Dubois et al. present a sufficient condition for coherence of two MISO rules. The proposition says that, if there exists at least one $i \in \{1, \dots, n\}$, such that two SISO rules $A_{1i} \rightarrow B_1, A_{2i} \rightarrow B_2$ are coherent, then two MISO rules are coherent as well. A_{1i}, A_{2i} are considered to be the components of the respective antecedents A_1, A_2 according to formula (10). The authors point out that this is only a sufficient condition which is not necessary.

Comparing this result with our, we can say that our sufficient condition is more specific than the one presented in Proposition 4.8. As an example, consider two rules of the Gaussian radial I-FS, see Section 5.5.2, with the parameters presented in Table 1. The rules are pairwise coherent according to Theorem 6, but they are not coherent according to Proposition 4.8 in [7].

	a_{j1}	a_{j2}	b_{j1}	b_{j2}	c_j	d_j	s_j
$j = 1$	0	0	1	1	0	1	0
$j = 2$	3	4	1	1	5	1	0

Table 1: Parameters of the Gaussian radial I-FS.

Proposition 4.10 states that the I-FS is coherent if and only if its rules are pairwise coherent. This is reflected by our Theorem 6 because checking the inequality (55) corresponds to the coherence test for the pairs of rules $j, k \in \{1, \dots, m\}$. That is, if (55) holds, then we are certain that the intersection $I_j^+(\mathbf{x}) \cap I_k^+(\mathbf{x})$ is non-empty for any input $\mathbf{x} \in X$. Since intervals $I_j^+(\mathbf{x})$ are subsets of intervals $I_j(\mathbf{x})$, see Definition 28, we are assured by the validity of (55) that $I_j(\mathbf{x}) \cap I_k(\mathbf{x}) \neq \emptyset$ for any input $\mathbf{x} \in X$. If this holds for all possible pairs of rules, we can imply the validity of (53), which gives the coherence of the radial I-FS. This actually corresponds to the sufficiency part of Proposition 4.10.

Discussing the necessary condition for coherence, we point out the following two observations. In fact, there are two sources of why (55) is not the necessary condition as well. First is the assertion of Lemma 9 that cannot be generally reversed. However, there is the special case when this can be done. It is the case when the radial I-FS employs the *act* function of the (ib) type in Definition 25. The Gaussian radial I-FS is an example of such the system. Then $I_j^+(\mathbf{x}) = I_j(\mathbf{x})$ for all j and $\mathbf{x} \in X$; and the non-emptiness of the intersection (54) is also necessary for coherence of the system.

The second source of sufficiency can be identified in the proof of Theorem 6. Namely, it is the replacement of $J_{\mathbf{b}_{jk}}(\mathbf{x})$ by $J_{jk}(\mathbf{x})$. If we did not take this replacement, then the inequality (60) would read

$$|c_j - c_k| - (s_j + s_k) > \min_{\mathbf{x} \in X} \{J_{\mathbf{b}_{jk}}(\mathbf{x})\}.$$

The problem here is with an analytic specification of the minimum of $J_{\mathbf{b}_{jk}}(\mathbf{x})$ that must be searched numerically. On the other hand, $J_{\mathbf{b}_{jk}}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$ forms a convex function and therefore numerical algorithms are effective here.

To end the discussion, we state that the paper [7] presents algorithms for checking the coherence of two rules with the trapezoidal fuzzy sets, but these are rather complicated and specific. Our Theorem 6 is general with respect to the entire class of the radial implicative fuzzy systems. In fact, we have enhanced the results and methodology presented in [7] mainly in practical applicability.

8. Conclusions

In the paper, we have introduced the class of the radial fuzzy systems. The class consists of the radial conjunctive and the radial implicative fuzzy systems. Computational models of the radial fuzzy systems draw on the notion of the radial MISO rule base and the standard configuration of the fuzzy system. The standard configuration comprises the singleton fuzzifier, the MISO rule base, the CRI inference engine and the WA or MOM method of defuzzification. The WA method is used in the conjunctive systems and the MOM method in the implicative ones.

In the radial rule bases, fuzzy sets are implemented by the radial functions; and the antecedents of rules exhibit the radial property which is a kind of the shape preservation property. In the paper, we have proved two theorems that enable building the radial fuzzy systems on the basis of the minimum t -norm and the continuous Archimedean t -norms.

Radial fuzzy systems compute functions according to their computational models. We have specified these models explicitly and showed that the radial property enables their convenient representations. With respect to the computational models, the notions of completeness and coherence were recalled. Completeness resides on the notion of the degree of covering (DOC) of the rule base. The DOC corresponds to the lower bound on maximum of degrees of firing of individual rules when going across of the input set of the fuzzy system.

Completeness ($\text{DOC} > 0$) is important not only to have the computational models of the radial fuzzy systems specified, but it may also be shown that the DOC makes a lower bound when comparing the inference engine's output fuzzy sets from the radial conjunctive and implicative fuzzy systems [3]. We did not present here an analysis of how the DOC can be computed for the radial rule bases. We only note that this task falls into the area of computational geometry and is also related to the problem of redundant rules detection [7, 8].

The other important notion is coherence. First of all, remark that coherence is not affected by the used defuzzification method. It relates to the core of the inference engine's output fuzzy set. We have shown that for the radial implicative fuzzy systems coherence can be assured by checking a rather simple sufficient condition. The condition has the form of inequalities between the parameters of the IF-THEN rules that constitute the rule base of the system. There is also the natural question on the necessary condition on coherence. In fact, this is related to the global minimum of the $J_{b_{jk}}(\mathbf{x})$ function that was introduced in the proof of Theorem 6. This minimum must be searched numerically, but the $J_{b_{jk}}$ function is convex so local minima are also global minima and numerical optimization works effectively here.

We see several application domains of the radial fuzzy systems. First, the input and output sets may be any normed spaces, hence elements of X may be rather complex objects, e.g., text documents. If we are able to introduce norms on these spaces, the radial fuzzy systems can be used to represent functions from X to Y . The implicative radial fuzzy systems then deliver a logical structure into the functions they represent.

Second, the radial fuzzy systems enable us to establish a bridge between the data driven descriptions of relations that correspond to the conjunctive systems and the logically driven descriptions that correspond to the implicative systems. In fact, if we omit denominator in the WA method of defuzzification, then we will find that computation of the radial C-FS corresponds to the computation of a RBF neural network [10]. The Gaussian systems are of priority interest here [12, 13]. Hence, there is an interesting field of future research that deals with fusion of data driven learning algorithms such as back-propagation and logical descriptions of relations. Moreover, using the kernel methods popular in machine learning [16] might be fruitful in the context of the radial fuzzy systems.

To sum up the novelties presented in the paper, we mention especially the actual introduction of the class of the radial fuzzy systems. The class accommodates the fuzzy systems which are commonly used in applications [29]. Recall the Mamdani conjunctive fuzzy system of Section 5.5.1 that employs the triangular fuzzy sets and the minimum t -norm. The Gaussian fuzzy sets and the product t -norm are also broadly used in applications and the Gaussian conjunctive radial fuzzy system represents a popular choice in practice [26, 12]. Hence the class as a whole does not only represent a convenient theoretical concept, but it has also a strong link to real-world problems. Recall for instance the car navigation example of Section 2.

Furthermore, a broad variety of other members of the class can be introduced drawing on different Archimedean t -norms, see Appendix in [14] for a comprehensive list of various parameterized families of t -norms. Each continuous Archimedean t -norm generates a radial rule base and consequently a conjunctive or implicative radial fuzzy system. Hence an user can design the radial fuzzy system that fits best her or his requirements. All such systems exhibit the radial property and therefore admit a unified approach for analyzing their properties such as completeness and/or coherence of their rule bases.

Coherence is the most important from the properties of the radial implicative fuzzy systems that we have analyzed in the paper. It corresponds to non-contradictoriness of knowledge stored in an implicative fuzzy system. It has been studied in different contexts in literature, e.g. in [28, 17, 22] and most thoroughly in [7], however, there has not been yet introduced a comfortable unifying view for its analysis. The framework of the radial fuzzy systems enables establishing such a view due to Theorem 6 that state the sufficient condition for testing coherence of any radial implicative fuzzy system. This makes a great benefit because the user is liberated from analyzing properties of individual fuzzy systems determined by individual shapes of the employed fuzzy sets.

Finally, we mention that the entire framework of the radial fuzzy systems seems promising to bring more interesting results in several other directions. These are related to the questions on universal approximation capabilities of the radial fuzzy systems, detection of redundant rules in radial rule bases and connection to the radial basis neural networks to mention a few. We are going to focus on these directions in our future research.

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